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SUBDIVIDED SHELL FLOWER GRAPHS: ρ - LABELING

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Abstract: A ρ -labeling (or ρ - valuation) of a graph is an injection from the vertices of the graph with 'q' edges to the set $\{0, 1, 2, 3, ..., (2q-1), 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, a_3, ..., a_{q-1}, a_q$ then $a_i = i$ or $a_i = (2q+1-i)$. A shell graph, $C(n; n-3)$, is defined as a cycle C_n with (n - 3) chords sharing a common endpoint called the apex. In other words, a shell graph is the join of complete graph K_1 and P_m, the path with m vertices. A subdivided shell graph is obtained from the shell graph $G = P_m \vee K_1$ by subdividing the edges in the path P_m of the shell graph. A subdivided shell flower graph is defined as a one vertex union of k copies of the subdivided shell graph and k copies of the complete graph K_2. In this paper, we prove that subdivided shell flower graphs admit ρ -labeling.

Keywords and Phrases: Shell graph, subdivided shell graph, subdivided shell flower graph, ρ - labeling.

2010 Mathematics Subject Classification: 05C78.

1. Introduction

A graph labeling is an assignment of non negative integers to the vertices and edges of the graph subject to certain conditions. In 1967, Rosa [7] introduced four types of labelings which includes ρ - labeling. A ρ - labeling of a graph is an injection from the vertices of the graph with q edges to the set$\{0, 1, 2, 3, ..., (2q-1), 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, a_3, ..., a_{q-1}, a_q$ then $a_i = i$ or $a_i = (2q+1-i)$.

Rosa [7] proved that a cyclic decomposition of the edge set of the complete graph K_{2q+1} into subgraphs isomorphic to a given graph G with 'q' edges exists if and only if G has a ρ -labeling [7], Donovan, El-Zanati, Vanden Eyden, and Sutinuntopas prove that ${}^{r}C_m$ has a ρ - labeling when $r \leq 4$ [5]. They conjecture that every 2 - regular graph has a ρ - labeling. Gannon and El-Zanati proved that for any odd n $\geq$ 7, ${}^{r}C_n$ admits ρ labelings [6]. Dinitz and Rodney proved the above result when $n = 3$ [2]. El-Zanati and Gannon proved that ${}^{r}C_n$ admits ρ - labeling when $n = 5$ [4]. For an exhaustive survey of graph labelings, one may refer to Gallian [5].

Shell graph was introduced by Deb and Limaye [1]. They define a shell graph as a cycle C_n with (n- 3) chords sharing a common end point called the apex. In other words, shell graph is the join of a complete graph K_1 and P_m, the path with m vertices. It is denoted as $C(n; n-3)$ (Figure 1 a). A subdivided shell graph is obtained from the shell graph $G = P_m \vee K_1$ by subdividing the edges in the path P_m of the shell graph (Figure 1 b).

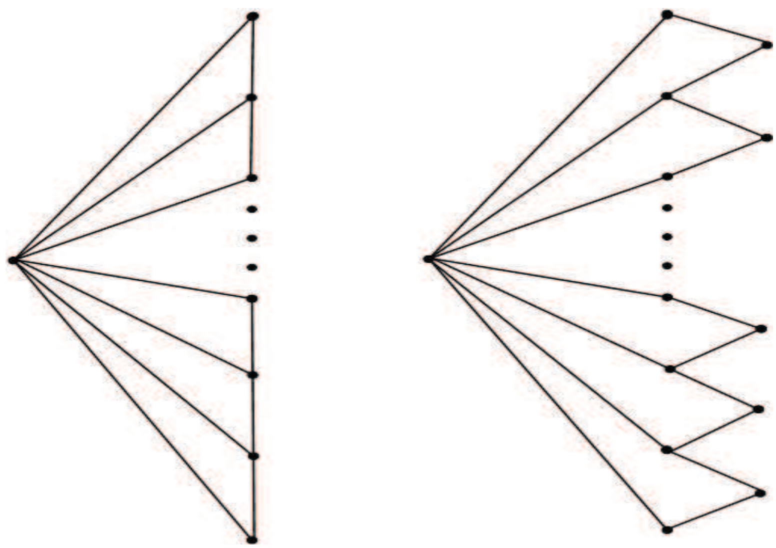

Figure 1 a. Shell graph **b. Subdivided shell graph**

A subdivided shell flower graph is defined as a one vertex union of k copies of the subdivided shell graph and k copies of the complete graph K_2. The subdivided shell flower graph is shown in graph is shown in Figure 2.

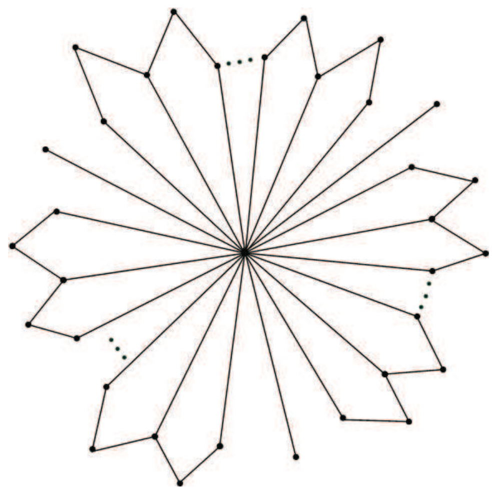

Figure 2 Subdivided shell flower graph when $k = 3$

Shell graphs have many applications in various fields. Subdivided shell flower graphs, being one vertex union of many subdivided shell graphs, have numerous applications in the field of X ray crystallography, Radar Communications and Networks, Secret Sharing Scheme, Coding theory and so on.

2. Main Result

In this section, we prove that subdivided shell flower graphs admit ρ - labeling.

Theorem : *Subdivided shell flower graphs, one vertex union of k copies of the subdivided shell graph and k copies of the complete graph K_2 admit ρ - labeling for $k = 3$.*

Proof. Let G be a subdivided shell flower graph with n vertices and q edges. Denote the apex of G as v_0. The number of path vertices in each subdivided shell is denoted as $m(m \geq 5)$. Denote the vertices in the path of the first subdivided shell as $v_1, v_2, \ldots, v_m$. The vertices in the path of the second subdivided shell are denoted as $v_{m+1}, v_{m+2}, \ldots, v_{2m}$. The vertices in the path of the third subdivided shell of G are denoted as $v_{2m+1}, v_{2m+2}, \ldots, v_{3m}$. Let u_1, u_2, u_3 be the end vertices in the pendant edges. Note that, G has $q = \frac{9m+3}{2}$ edges and $n = (3m + 4)$ vertices.

The theorem is proved in four Cases: when m is congruent to -3, -1, 1, 3 (mod 8).

Define $f : V(G) \rightarrow \{0, 1, 2, 3, ..., 2q\}$ as follows.

$$f(v_0) = 0$$

$$f(v_{2i-1}) = \begin{cases} 9m-i+4, & \text{for } 1 \leq i \leq \frac{(m+1)}{2} \\ \frac{9m+2i-1}{2}, & \text{for } \frac{(m+3)}{2} \leq i \leq m \\ 9m-i+3, & \text{for } (m+1) \leq i \leq \frac{(3m+1)}{2} \end{cases}$$

$$f(u_1) = \frac{7m+5}{2}$$

$$f(u_2) = \frac{5m+5}{2} \qquad (1)$$

In all the four Cases, the above vertex labels are one and the same. The label of the third pendant vertex u_3 and the labels of even suffix vertices v_{2i} of G differ in each Case. They are given below.

Case 1: When $(m+3) \equiv 0 \pmod 8$.
In this Case $m = 8j$ - 3 where $j \geq 1$. Define

$$f(v_{2i}) = \begin{cases} \frac{9m+2i+1}{2}, & \text{for } 1 \leq i \leq \frac{(m-1)}{2} \\ 9m-i+3, & \text{for } \frac{m+1}{2} \leq i \leq m \\ \frac{9m+2i-1}{2}, & \text{for } (m+1) \leq i \leq \frac{9m+3}{8} \\ \frac{4i-3m-5}{4}, & \text{for } \frac{9m+11}{8} \leq i \leq \frac{(3m-1)}{2} \end{cases}$$

$$f(u_3) = \frac{27m+13}{4} \qquad (2)$$

From equations (1) and (2), it is clear that all vertices are assigned labels and they are distinct. If any two vertices have same labels then a contradiction to the hypothesis will occur. For example, if$f(v_{2i-1})= f(v_{2i})$ for $1\leq i \leq \frac{m-1}{2}$, then we would get the value of 'm' as a fraction which is not true. This proves that the function f is an injection. Also$f(v) \in \{0, 1, 2, ..., 2q\}$.

The edge labels are computed as follows.

$$|f(v_0) - f(v_{2i-1})| = \begin{cases} |9m-i+4|, & \text{for } 1 \leq i \leq \frac{(m+1)}{2} \\ |9m-i+3|, & \text{for } (m+1) \leq i \leq \frac{(3m+1)}{2} \end{cases}$$

$$|f(v_0) - f(v_{2i})| = |9m-i+3| \qquad \text{for } \frac{m+1}{2} \leq i \leq m$$

$$|f(v_0) - f(u_1)| = \frac{7m+5}{2}$$

$$|f(v_0) - f(u_2)| = \frac{5m+5}{2}$$

$$|f(v_0) - f(u_3)| = \frac{27m+13}{4}$$

$$|f(v_{2i-1}) - f(v_{2i})| = \begin{cases} |\frac{9m-4i+7}{4}|, & \text{for } \begin{cases} 1 \le i \le \frac{m-1}{2} \ \& \\ \frac{m+3}{2} \le i \le m \ \& \\ (m+1) \le i \le \frac{9m+3}{8} \end{cases} \\ |\frac{39m-8i+17}{4}|, & \text{for } \frac{9m+11}{8} \le i \le \frac{3m-1}{2} \end{cases}$$

$$|f(v_{2i}) - f(v_{2i+1})| = \begin{cases} |\frac{9m-4i+5}{4}|, & \text{for } \begin{cases} 1 \le i \le \frac{m-1}{2} \ \& \\ \frac{m+1}{2} \le i \le (m-1) \ \& \\ (m+1) \le i \le \frac{9m+3}{2} \end{cases} \\ |\frac{39m-8i+13}{4}|, & \text{for } \frac{9m+11}{8} \le i \le \frac{3m-1}{2} \end{cases} \tag{3}$$

From the computations given in equations (3) it is evident that all the edge labels are distinct and they satisfy the condition $e_i = 2q + 1 - i$ when i ranges from 1 to $\frac{q}{2}$ and $e_i = i$ when i ranges from $\frac{q+2}{2}$ to q.

Case 2: When $(m + 1) \equiv 0 \pmod 8$.
In this Case $m = 8j$ - 1 where $j \ge 1$. Define

$$f(v_{2i}) = \begin{cases} \frac{9m+2i+1}{2}, & \text{for } 1 \le i \le \frac{m-1}{2} \\ 9m - i + 3, & \text{for } \frac{m+1}{2} \le i \le m \\ \frac{9m+2i-1}{2}, & \text{for } (m+1) \le i \le \frac{9m+1}{8} \\ \frac{4i-3m-3}{4}, & \text{for } \frac{9m+9}{8} \le i \le \frac{3m-1}{2} \end{cases}$$

$$f(u_3) = \frac{27m+11}{4} \tag{4}$$

From equations given in (1) and (4) it is clear that all vertices are assigned labels and they are distinct. If any two vertex labels are equal then a contradiction would occur to the hypothesis as in Case 1. Hence the function f is an injection. Also one can note that, $f(v) \in \{0, 1, 2, ..., 2q\}$.

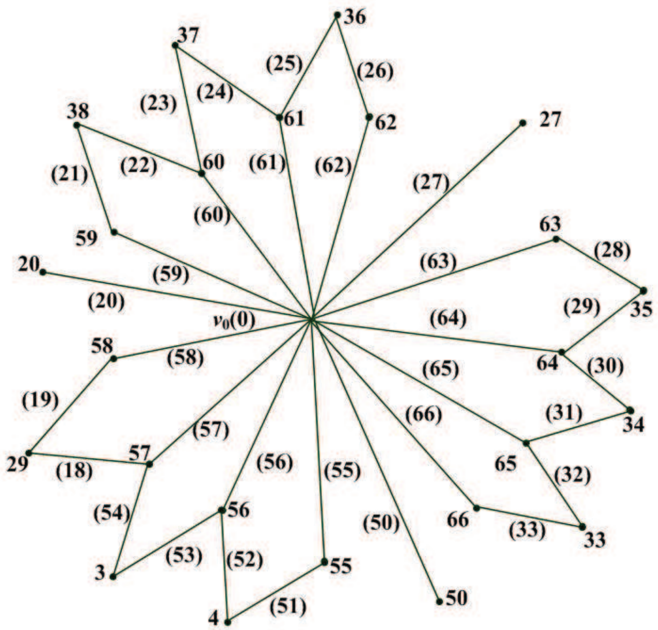

Figure 3 ρ -labeling in a subdivided shell flower graph when $m = 7$, $n = 25$, $q = 33$

The edge labels are computed as follows.

$$|f(v_0) - f(v_{2i-1})| = \begin{cases} |9m - i + 4|, & \text{for } 1 \le i \le \frac{(m+1)}{2} \\ |9m - i + 3|, & \text{for } (m+1) \le i \le \frac{(3m+1)}{2} \end{cases}$$

$$|f(v_0) - f(v_{2i})| = |9m - i + 3| \qquad for \ \frac{m+1}{2} \le i \le m$$

$$|f(v_0) - f(u_1)| = \frac{7m+5}{2}$$

$$|f(v_0) - f(u_2)| = \frac{5m+5}{2}$$

$$|f(v_0) - f(u_3)| = \frac{27m+11}{4}$$

$$|f(v_{2i-1}) - f(v_{2i})| = \begin{cases} |\frac{9m-4i+7}{4}|, & \text{for } \begin{cases} 1 \le i \le \frac{m-1}{2} \ \& \\ \frac{m+3}{2} \le i \le m \ \& \\ (m+1) \le i \le \frac{9m+1}{8} \end{cases} \\ |\frac{39m-8i+15}{4}|, & \text{for } \frac{9m+9}{8} \le i \le \frac{3m-1}{2} \end{cases}$$

$$|f(v_{2i})-f(v_{2i+1})| = \begin{cases} |\frac{9m-4i+5}{4}|, & \text{for } \begin{cases} 1 \le i \le \frac{m-1}{2} \ \& \\ \frac{m+1}{2} \le i \le (m-1) \ \& \\ (m+1) \le i \le \frac{9m+1}{8} \end{cases} \\ |\frac{39m-8i+11}{4}|, & \text{for } \frac{9m+9}{8} \le i \le \frac{3m-1}{2} \end{cases} \quad (5)$$

From the computations given in equations (5) it is evident that all the edge labels are distinct and they satisfy the conditions $e_i = 2q+1-i$ when i ranges from 1 to $\frac{q+1}{2}$ and $e_i = i$ when i ranges from $\frac{q+3}{2}$ to q.

An illustration for this Case is given in Figure 3 when $m = 7, n = 25, q = 33$.

Case 3: When $(m-1) \equiv 0(\text{mod } 8)$.

In this Case $m = 1+8j$ where $j \ge 1$. Define

$$f(v_{2i}) = \begin{cases} \frac{9m+2i+1}{2}, & \text{for } 1 \le i \le \frac{(m-1)}{2} \\ 9m-i+3, & \text{for } \frac{m+1}{2} \le i \le m \\ \frac{9m+2i-1}{2}, & \text{for } (m+1) \le i \le \frac{9m-1}{8} \\ \frac{4i-3m-1}{4}, & \text{for } \frac{9m+7}{8} \le i \le \frac{(3m-1)}{2} \end{cases}$$

$$f(u_3) = \frac{9m+7}{4} \quad (6)$$

In this Case also, it is obvious from equations (1) and (6) that all vertices are assigned labels and they are distinct. If any two vertex labels are equal then we get a contradiction to our hypothesis as in Case 1.

This proves that the function f is an injection. Also $f(v) \in \{0, 1, 2, ..., 2q\}$.

The edge labels computations are given below.

$$|f(v_0)-f(v_{2i-1})| = \begin{cases} |9m-i+4|, & \text{for } 1 \le i \le \frac{(m+1)}{2} \\ |9m-i+3|, & \text{for } (m+1) \le i \le \frac{(3m+1)}{2} \end{cases}$$

$$|f(v_0)-f(v_{2i})| = |9m-i+3| \qquad for \ \frac{m+1}{2} \le i \le m$$

$$|f(v_{2i-1})-f(v_{2i})| = \begin{cases} |\frac{9m-4i+7}{4}|, & \text{for } \begin{cases} 1 \le i \le \frac{m-1}{2} \ \& \\ \frac{m+3}{2} \le i \le m \ \& \\ (m+1) \le i \le \frac{9m-1}{8} \end{cases} \\ |\frac{39m-8i+13}{4}|, & \text{for } \frac{9m+7}{8} \le i \le \frac{3m-1}{2} \end{cases}$$

$$|f(v_{2i})-f(v_{2i+1})| = \begin{cases} |\frac{9m-4i+5}{4}|, & \text{for } \begin{cases} 1 \le i \le \frac{m-1}{2} \ \& \\ \frac{m+1}{2} \le i \le (m-1) \ \& \\ (m+1) \le i \le \frac{9m-1}{8} \end{cases} \\ |\frac{39m-8i-9}{4}|, & \text{for } \frac{9m+7}{8} \le i \le \frac{3m-1}{2} \end{cases}$$

$$|f(v_0)-f(u_1)| = \frac{7m+5}{2}$$

$$|f(v_0)-f(u_2)| = \frac{5m+5}{2}$$

$$|f(v_0)-f(u_3)| = \frac{9m+7}{4} \tag{7}$$

It is evident that all edge labels are distinct and they satisfy the conditions $e_i = 2q+1-i$ when i ranges from 1 to $\frac{q}{2}$ and $e_i = i$ when i ranges from $\frac{q+2}{2}$ to q.

Case 4: When $(m-3) \equiv 0(\text{mod } 8)$.

In this Case $m = 3+8j$ where $j \ge 1$. Define

$$f(v_{2i}) = \begin{cases} \frac{9m+2i+1}{2}, & \text{for } 1 \le i \le \frac{m-1}{2} \\ 9m-i+3, & \text{for } \frac{m+1}{2} \le i \le m \\ \frac{9m+2i-1}{2}, & \text{for } (m+1) \le i \le \frac{9m-3}{8} \\ \frac{4i-3m+1}{4}, & \text{for } \frac{9m+5}{8} \le i \le \frac{3m-1}{2} \end{cases}$$

$$f(u_3) = \frac{9m+9}{4} \tag{8}$$

From the definitions it is clear that no two vertex labels are the same. Also, we can see that $f(v) \in \{0, 1, 2, ..., 2q\}$. The edge labels are computed as follows.

$$|f(v_0)-f(v_{2i-1})| = \begin{cases} |9m-i+4|, & \text{for } 1 \le i \le \frac{(m+1)}{2} \\ |9m-i+3|, & \text{for } (m+1) \le i \le \frac{(3m+1)}{2} \end{cases}$$

$$|f(v_0)-f(v_{2i})| = |9m-i+3| \qquad for \ \ \frac{m+1}{2} \le i \le m$$

$$|f(v_0)-f(u_1)| = \frac{7m+5}{2}$$

$$|f(v_0)-f(u_2)| = \frac{5m+5}{2}$$

$$|f(v_0)-f(u_3)| = \frac{9m+9}{4}$$

$$|f(v_{2i-1}) - f(v_{2i})| = \begin{cases} |\frac{9m-4i+7}{4}|, & \text{for } \begin{cases} 1 \leq i \leq \frac{m-1}{2} \ \& \\ \frac{m+3}{2} \leq i \leq m \ \& \\ (m+1) \leq i \leq \frac{9m-3}{8} \end{cases} \\ |\frac{39m-8i+11}{4}|, & \text{for } \frac{9m+5}{8} \leq i \leq \frac{3m-1}{2} \end{cases}$$

$$|f(v_{2i}) - f(v_{2i+1})| = \begin{cases} |\frac{9m-4i+5}{4}|, & \text{for } \begin{cases} 1 \leq i \leq \frac{m-1}{2} \ \& \\ \frac{m+1}{2} \leq i \leq (m-1) \ \& \\ (m+1) \leq i \leq \frac{9m-3}{8} \end{cases} \\ |\frac{39m-8i+7}{4}|, & \text{for } \frac{9m+5}{8} \leq i \leq \frac{3m-1}{2} \end{cases} \quad (9)$$

The above computations show that all the edge labels are distinct and they satisfy the ρ -labeling conditions $e_i = 2q+1-i$ when i ranges from 1 to $\frac{q+1}{2}$ and $e_i = i$ when i ranges from $\frac{q+3}{2}$ to q. Thus, in all the Cases the function $'f'$ is shown to be an injection. From the edge label computations, it is evident that they are all distinct and it satisfies the conditions of ρ -labeling . Hence all subdivided shell flower graphs, when $k = 3$, admit ρ -labeling.

3. Conclusion

In this paper we have proved that subdivided shell flower graphs, one vertex union of k copies of the subdivided shell graph and k copies of the complete graph K_2, admit ρ - labeling. This result can be extended to all k. One can try to find out whether these graphs admit other labelings also.

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