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ON SOME SIGNED GRAPHS OF FINITE GROUPS

R. Rajendra, P. Siva Kota Reddy* and V. M. Siddalingaswamy**

Department of Mathematics,
Mangalore University, Mangalagangothri,
Mangaluru -574 199, INDIA.
E-mail: rrajendrar@gmail.com

*Department of Mathematics
Siddaganga Institute of Technology
B. H. Road, Tumkur-572 103, INDIA.
E-mail: reddy_math@yahoo.com; pskreddy@sit.ac.in

**Department of Mathematics
J. S. S. Academy of Technical Education
Uttarahalli-Kengeri Main Road, Bangalore-560 060, INDIA.
E-mail: swamyvms@yahoo.com

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Abstract: In this paper, we define two signed graphs namely, the order prime signed graph $OPS(\Gamma)$ and the general order prime signed graph $GOPS(\Gamma)$ of a given finite group Γ of order n. We discuss some properties of these two signed graphs.

Keywords and Phrases: Graph, group, order prime graph, general order prime graph, signed graph, order prime signed graph, general order prime signed graph.

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1. Introduction

For standard terminology and notion in group theory and graph theory, we refer the reader to the text-books of Herstein [3] and Harary [1] respectively. The non-standard will be given in this paper as and when required.

Throughout this paper, Γ denotes a finite group and the group of residue classes modulo n is denoted by $\mathbb{Z}_n$. The order of an element a in a group Γ is denoted by $o(a)$ and order of Γ is denoted by $o(\Gamma)$. The greatest common divisor (gcd) of two numbers x and y is denoted by (x, y).

In [5], M. Sattanathan and R. Kala defined the order prime graphs of finite groups and studied some properties of order prime graphs. In [4], we have defined the general order prime graphs of finite groups and studied some properties.

We have concentrated on the commuting property of elements in finite non-abelian groups to define signed graphs associated with a finite group. In this paper, we define order prime signed graph and general order prime signed graph of a finite group and discuss some properties of these two signed graphs. We recall the definitions of order prime graph, general order prime graph and signed graph.

Definition 1.1. *[5] The order prime graph $OP(\Gamma)$ of a finite group Γ of order n is defined as a graph with the vertex set $V(OP(\Gamma)) = \Gamma$ and two vertices a and b are adjacent in $OP(\Gamma)$ if and only if $(o(a), o(b)) = 1$.*

Definition 1.2. *[4] The general order prime graph $GOP(\Gamma)$ of a given finite group Γ of order n is defined as a graph with vertex set $V(GOP(\Gamma)) = \Gamma$ and any two vertices a and b are adjacent in $GOP(\Gamma)$ if and only if $(o(a), o(b)) = 1$ or p, where p is a prime and $p < n$.*

Note: We do not consider self-loops in $GOP(\Gamma)$ though in some cases we have, for some $a \in \Gamma$, $(o(a), o(a)) = 1$ or a prime p, $p < n$.

Definition 1.3. *[2,6] A signed graph is an ordered pair $S = (G, \sigma)$, where $G = (V, E)$ is a graph called underlying graph of S and $\sigma : E \to \{+, -\}$ is a function.*

A signed graph $S = (G, \sigma)$ is balanced if every cycle in S has an even number of negative edges [2]. Equivalently, a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

2. Order prime signed graphs

Definition 2.1. *The order prime signed graph $OPS(\Gamma)$ of a finite group Γ is the signed graph $((OP(\Gamma), \sigma)$ where the function $\sigma : E(OP(\Gamma)) \to \{+, -\}$ is given by*

$$\sigma((a,b)) = \begin{cases} +, & \text{if } ab = ba; \\ -, & \text{otherwise.} \end{cases}$$

By the Definition 2.1, it is obvious that, if a group Γ is abelian then all the edges of $OPS(\Gamma)$ are of '+' sign and in this case $OPS(\Gamma)$ is a balanced signed graph. If $OPS(\Gamma)$ contains atleast one edge with '−' sign, then the group Γ is non-abelian. But the converse of this statement is not true in general. For non-abelian groups of prime power order all the edges of $OPS(\Gamma)$ will be of '+' sign.

Example 2.2. *Consider the group $\mathbb{Z}_6$ under the operation addition modulo 6. The* *corresponding order prime signed graph $OPS(\mathbb{Z}_6)$ is shown in Fig 2.1.*

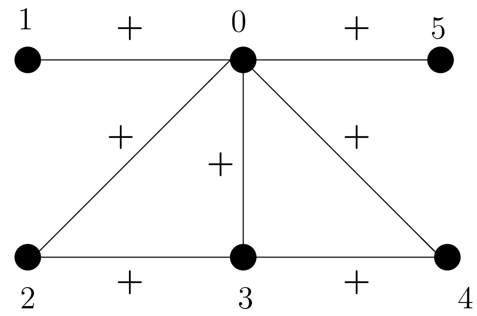

Fig. 2.1. $OPS(\mathbb{Z}_6)$

Example 2.3. *Consider the permutation group S_3 of 3 symbols. The corresponding order prime signed graph is shown in Fig 2.2.*

$$S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$$

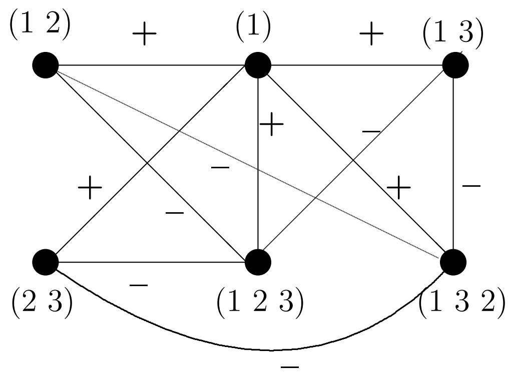

Fig. 2.2. $OPS(S_3)$

Remark: Order prime signed graph of a group need not always be a triangulated signed graph and also it need not be balanced.

The following results obtained for order prime signed graphs are anologous to the results concerning order prime graphs [5].

Proposition 2.4. *If Γ is a group of order n, then $OPS(\Gamma)$ is a connected signed graph and the maximum positive degree $\Delta^+(OPS(\Gamma)) = n - 1$.*

Proposition 2.5. *For any group Γ, the signed graph $OPS(\Gamma)$ is a complete graph with edges assigned '+' sign if and only if $o(\Gamma) = 2$.*

Proposition 2.6. *For any group Γ, the signed graph $OPS(\Gamma)$ can never be a unicyclic graph.*

Notation: We denote a signed graph (G, σ) with all edges assigned '+' sign by G^+.

Theorem 2.7. *If Γ is a finite group of order $n = p^\alpha$ where p is a prime number and $\alpha \in \mathbb{Z}^+$, then $OPS(\Gamma) \cong K^+_{1,n-1}$.*

Corollary 2.8. *Let Γ be a finite group of order n. Then the signed graph $OPS(\Gamma)$ is a tree with all its edges assigned '+' sign if and only if $n = p^\alpha$, where p is a prime number and $\alpha \in \mathbb{Z}^+$.*

Remark: If Γ is an abelian group with order $o(\Gamma) = p^\alpha$ where p is a prime number $\alpha \in \mathbb{Z}^+$, then $OPS(\Gamma)$ is a tree with all its edges assigned '+' sign. But the converse of this statement is not true, because abelian and non-abelian groups of same order p^α $(\alpha > 2)$ have the same order prime signed graph, that is a tree $K^+_{1,p^\alpha-1}$ with all its edges assigned '+' sign, because the identity element e commutes with every element of Γ.

Proposition 2.9. *Let Γ be a finite cyclic group. Then $OPS(\Gamma)$ is a signed graph with all its edges assigned '+' sign and has at least two pendent vertices.*

Theorem 2.10. *If Γ_1, Γ_2 are two groups such that $\Gamma_1 \cong \Gamma_2$, then $OPS(\Gamma_1) \cong OPS(\Gamma_2)$.*

Converse of the above Theorem 2.10 is not true in general. For, consider the groups $\mathbb{Z}_4$ and K_4. Note that $OPS(\mathbb{Z}_4) \cong OPS(K_4) \cong K^+_{1,3}$, but $\mathbb{Z}_4$ and K_4 are not isomorphic.

Theorem 2.11. *Let Γ be a group. Then $Aut(\Gamma) \subseteq Aut(OPS(\Gamma))$.*

Note: The converse of the Theorem 2.11 is not true in general.

Theorem 2.12. *Let Γ be a group. Suppose that the signed graph $OPS(\Gamma)$ has two adjacent vertices a, b such that $o(a)o(b) = o(\Gamma)$. Then the set $\{a, b\}$ is a generating set of Γ. Moreover, if the edge (a, b) has '+' sign if and only if Γ is abelian.*

Theorem 2.13. *Let Γ be an abelian group. Let $X \subseteq \Gamma$ be such that the graph induced by X is complete (in the sense of unsigned graph) in the signed graph $OPS(\Gamma)$ and the product of the order of all elements of X is same as of $o(\Gamma)$. Then X is a generating set of Γ.*

Theorem 2.14. *Let Γ be a group with $o(\Gamma) = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where p_i's are prime numbers and $n_i \in \mathbb{Z}^+$, $(1 \leq i \leq k)$. Then the signed graph $OPS(\Gamma)$ is a complete*

$(k+1)$-partite graph (in the sense of unsigned graph) if and only if $o(a) = p_i^j$, $\forall a \in \Gamma - \{e\}$, $1 \leq i \leq k$ *and* $1 \leq j \leq n_i$.

3. General order prime signed graphs

Definition 3.1. *The order prime signed graph* $GOPS(\Gamma)$ *of a finite group* Γ *is the signed graph* $((GOP(\Gamma), \sigma)$ *where the function* $\sigma : E(GOP(\Gamma)) \to \{+, -\}$ *is given by*

$$\sigma((a,b)) = \begin{cases} +, & \text{if } ab = ba; \\ -, & \text{otherwise.} \end{cases}$$

By the Definition 3.1, it is obvious that, if a group Γ is abelian then all the edges of $GOPS(\Gamma)$ are of '+' sign and so in this case $GOPS(\Gamma)$ is a balanced signed graph. If $GOPS(\Gamma)$ contains atleast one edge with '$-$' sign, then the group Γ is non-abelian.

By the Definitions 2.1 and 3.1, it follows that for any finite group Γ, $OPS(\Gamma)$ is a subgraph of $GOPS(\Gamma)$. Since the identity element e is the only element of order 1 in any group Γ of order n, it follows that, the graph $OPS(\Gamma)$ and $GOPS(\Gamma)$ are connected. Also, $d^+(e) = n-1$ and the maximum positive degree $\Delta^+(GOPS(\Gamma)) = \Delta^+(OPS(\Gamma)) = n-1$.

Proposition 3.2. *If* Γ *is a group of prime order then* $GOPS(\Gamma) = OPS(\Gamma)$.

Proof. Suppose that $o(\Gamma)$ is a prime number p. Then Γ is an abelian group and so all edges in $GOPS(\Gamma)$ and $OPS(\Gamma)$ are assigned '+' sign. Since the positive divisors of a prime p are 1 and p itself, and by the definitions of $GOPS(\Gamma)$ and $OPS(\Gamma)$, it follows that, $GOPS(\Gamma) = OPS(\Gamma)$.

Theorem 3.3. *If* Γ *is a finite cyclic group and* $OPS(\Gamma) = GOPS(\Gamma)$, *then* $o(\Gamma)$ *is a prime number.*

Proof. Suppose Γ is a finite cyclic group and $OPS(\Gamma) = GOPS(\Gamma)$. We claim that $o(\Gamma)$ is a prime number. If $o(\Gamma) = n$ is not a prime, then there exists a prime number p $(p < n)$ dividing $o(\Gamma) = n$ and by the Cauchys theorem for finite groups, Γ has an element a of order p. Since Γ is cyclic, there exists an element b in Γ with $o(b) = n$. Now $(o(a), o(b)) = (p, n) = p$. Therefore a and b are adjacent in $GOPS(\Gamma)$. But $GOPS(\Gamma) = OPS(\Gamma)$ and so p must be equal to 1, which is a contradiction. Hence n is a prime number.

By virtue of the Proposition 3.2 and the Theorem 3.3, we have the following corollary:

Corollary 3.4. *An integer* $n > 1$ *is a prime number if and only if* $OPS(\mathbb{Z}_n) = GOPS(\mathbb{Z}_n)$.

Theorem 3.5. *Let Γ be a finite group of order n. Then $GOPS(\Gamma) \cong K^{+}_{1,n-1}$ if and only if $o(\Gamma) = n$ is a prime number.*

Proof. Suppose that $o(\Gamma) = p$ is a prime number. Then Γ is a cyclic group. Being a cyclic group, Γ is abelian and so all the edges in $GOPS(\Gamma)$ are assigned '+' sign. Let $\Gamma = \{e, a, a^2, \ldots, a^{p-1}\}$, where e is the identity element and a is a generator of the group Γ. Note that in the group Γ, $o(e) = 1$ and $o(a^i) = p$, $1 \leq i \leq p-1$. Hence $(o(e), o(a^i)) = 1$ and $(o(a^i), o(a^j)) = p$, $1 \leq i, j \leq p-1$. Therefore e is adjacent to a^i for all $i = 1, 2, \ldots, p-1$ and a^is are mutually non-adjacent in $GOPS(\Gamma)$ and hence $GOPS(\Gamma) \cong K^{+}_{1,p-1}$.

Conversely, suppose that $GOPS(\Gamma) \cong K^{+}_{1,n-1}$. Clearly, $GOP(\Gamma) - e$ is totally disconnected. We claim that $o(\Gamma) = n$ is a prime number. If n is not a prime number, then there exists a prime p dividing n. Since $p \mid n$, by the Cauchy's theorem for finite groups, there exists an element a in Γ such that $o(a) = p$. Now, for any element $x \neq e$ in Γ, $(o(a), o(x)) = 1$ or p. Hence a and x are adjacent in $GOP(\Gamma) - e$, which is a contradiction and so n is a prime number.

The following corollaries are immediate from the Theorem 3.5.

Corollary 3.6. *Let Γ be a finite group of order n. Then $GOPS(\Gamma)$ is a tree (in which all edges are assigned '+' sign) if and only if $o(\Gamma)$ is a prime number.*

Corollary 3.7. *An integer $n > 1$ is prime if and only if $GOPS(\mathbb{Z}_n)$ is a tree (in which all edges are assigned '+' sign).*

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