South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752 (Print)**
Vol. 14, No. 3 (2018), pp. 31-48

INVERSE SUM CONNECTIVITY ENERGY OF A GRAPH

M. J. Sridevi and K. R. Rajanna*

Department of Mathematics,
Governament Science College Hassan-573201, INDIA
E-mail: sridevigsch@gmail.com

*Department of Mathematics,
Acharya Institute of Technology
Soladevanahally Bengaluru-560107, INDIA

(Received: July 8, 2018)

Abstract: In this paper we introduce and investigate the inverse sum connectivity energy of a graph $ISCE(G)$. We establish upper and lower bounds for $SDDE(G)$. Also the symmetric division deg energy for certain graphs with one edge deleted are calculated.

Keywords and Phrases: inverse sum connectivity index, inverse sum connectivity eigenvalues, inverse sum connectivity energy, k-complement, k(i)-complement, edge deletion.

2010 Mathematics Subject Classification: 05C50.

1. Introduction

Let G be a simple graph and let $\{v_1, v_2, \cdots, v_n\}$ be the set of its vertices. Let $i, j \in \{1, 2, \cdots, n\}$. If two vertices v_i and v_j of G are adjacent, then we use the notation $v_i \sim v_j$. For a vertex $v_i \in V(G)$, the degree of v_i will be denoted by $d(v_i)$ or briefly by d_i.

In chemistry, topological indices play an important role due to their numerous applications. There are many topological indices such as Randić index, sum-connectivity index, atom bond connectivity index, Zagreb indices, etc. Inverse sum connectivity is a new molecular descriptor, introduced by K.N Prakasha [4]. He defined the new index of a graph G as follows

$$ISCM(G) = ISC(G) = \sum_{i \sim j} \left(\sqrt{d_i + d_j} \right).$$

By the motivation of the inverse sum connectivity index, we associate a symmetric square matrix $ISC(G)$ to a graph G. The inverse sum connectivity matrix $ISC(G) = (I_{ij})_{n\times n}$ is defined as

$$I_{ij} = \begin{cases} \sqrt{d_i + d_j} & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

2. The inverse sum connectivity energy of a graph

Through out the paper we considered a graph G which is a simple, finite, undirected, sign-less, mark-less graph. The energy $E(G)$ is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. For more details on energy of a graph, refer [2,3].

Let $ISC(G)$ be the inverse sum connectivity matrix. The characteristic polynomial of $ISC(G)$ will be denoted by $\phi_{ISCD}(G, \lambda)$ and defined as

$$\phi_{ISC}(G, \lambda) = det(\lambda I - ISC(G)).$$

Since the inverse sum connectivity matrix is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The inverse sum connectivity energy of G is similarly defined by

$$ISCE(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

This paper is organized as follows. In Section 3, we give the basic properties of inverse sum connectivity energy of a graph. In Section 4, inverse sum connectivity energy of some standard and well studied graphs are obtained. In Section 5, we compute inverse sum connectivity energy of some complements of some specific graphs. In Section 6, the inverse sum connectivity energy for certain graphs with one edge deleted are calculated and finally in Section 7, some open problems are given.

3. Some basic properties of inverse sum connectivity energy of a graph

Let us define the number P as

$$J = \sum_{i<j} \sqrt{(d_i + d_j)}.$$

Then we have

Proposition 3.1. *The first three coefficients of the polynomial $\phi_{ISC}(G, \lambda)$ are as follows:*

(i) $a_0 = 1$,

(ii) $a_1 = 0$,

(iii) $a_2 = -J$.

Proof. (i) By the definition of $\Phi_{ISC}(G, \lambda) = det[\lambda I - ISC(G)]$, we get $a_0 = 1$. (ii) The sum of determinants of all 1×1 principal submatrices of $ISC(G)$ is equal to the trace of $ISC(G)$ implying that

$$a_1 = (-1)^1 \times \text{the trace of } ISC(G) = 0.$$

(iii) By the definition, we have

$$\begin{aligned}(-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\ &= -J.\end{aligned}$$

Proposition 3.2. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the inverse sum connectivity eigenvalues of $ISC(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2J.$$

Proof. It follows as

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ii})^2 \\ &= 2\sum_{i<j} (a_{ij})^2 \\ &= 2J.\end{aligned}$$

Using this result, we now obtain lower and upper bounds for the inverse sum connectivity energy of a graph:

Theorem 3.3. *Let G be a graph with n vertices. Then*

$$ISC(G) \leq \sqrt{2nJ}.$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $ISC(G)$. By the Cauchy-Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

Let $a_i = 1$, $b_i =\mid \lambda_i \mid$. Then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

implying that

$$[ISCE]^2 \leq n \cdot 2J$$

and hence we get

$$[ISCE] \leq \sqrt{2nJ}$$

as an upper bound.

Theorem 3.4. *Let G be a graph with n vertices. If R= det $ISC(G)$, then*

$$ISCE(G) \geq \sqrt{2J + n(n-1)R^{\frac{2}{n}}}.$$

Proof. By definition, we have

$$\begin{aligned}
(ISCE(G))^2 &= \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \\
&= \sum_{i=1}^{n} \mid \lambda_i \mid \sum_{j=1}^{n} \mid \lambda_j \mid \\
&= \left(\sum_{i=1}^{n} \mid \lambda_i \mid^2\right) + \sum_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid .
\end{aligned}$$

Using arithmetic-geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid \;\geq\; \left(\prod_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}}.$$

Therefore,

$$\begin{aligned}
[ISC(G)]^2 &\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1)\left(\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}} \\
&\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1)\left(\prod_{i=1}^{n} \mid \lambda_i \mid^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
&= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1)R^{\frac{2}{n}} \\
&= 2J + n(n-1)R^{\frac{2}{n}}.
\end{aligned}$$

Thus,

$$ISCE(G) \geq \sqrt{2J + n(n-1)R^{\frac{2}{n}}}.$$

Let λ_n and λ_1 are the minimum and maximum values of all $\lambda_i's$. Directly we can obtain the following results.

Theorem 3.5. *For a graph G of order n,*

$$ISCE(G) \geq \sqrt{2Jn - \frac{n^2}{4}(\lambda_1 - \lambda_n)^2}.$$

Theorem 3.6. *For a graph G of order n with non-zero eigenvalues, we have*

$$ISCE(G) \geq \frac{2\sqrt{\lambda_1\lambda_n}\sqrt{2Jn}}{(\lambda_1 + \lambda_n)^2}.$$

Theorem 3.7. *Let G be a graph of order n. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n$ be the eigenvalues in increasing order. Then*

$$ISCE(G) \geq \frac{|\lambda_1||\lambda_n|n + 2J}{|\lambda_1| + |\lambda_n|}.$$

4. Inverse sum connectivity energy of some standard graphs

In this section, we calculate the inverse sum connectivity energy of some well-known and frequently used graph types including complete, cycle, star, friendship, cocktail party, double star, Dutch windmill, crown and complete bipartite graphs.

Theorem 4.1. *The inverse sum connectivity energy of a complete graph K_n is*

$$ISCE(K_n) = (2n-2)^{\frac{3}{2}}.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. For this graph, the inverse sum connectivity matrix is

$$ISC(K_n) = \begin{bmatrix} 0 & \sqrt{2n-2} & \sqrt{2n-2} & \dots & \sqrt{2n-2} & \sqrt{2n-2} \\ \sqrt{2n-2} & 0 & \sqrt{2n-2} & \dots & \sqrt{2n-2} & \sqrt{2n-2} \\ \sqrt{2n-2} & \sqrt{2n-2} & 0 & \dots & \sqrt{2n-2} & \sqrt{2n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{2n-2} & \sqrt{2n-2} & \sqrt{2n-2} & \dots & 0 & \sqrt{2n-2} \\ \sqrt{2n-2} & \sqrt{2n-2} & \sqrt{2n-2} & \dots & \sqrt{2n-2} & 0 \end{bmatrix}.$$

The characteristic equation then becomes

$$(\lambda + \sqrt{2n-2})^{n-1}(\lambda - (n-1)\sqrt{2n-2}) = 0$$

and the spectrum would be

$$Spec_{ISC}(K_n) = \begin{pmatrix} -\sqrt{2n-2} & n-1\sqrt{2n-2} \\ n-1 & 1 \end{pmatrix}.$$

Therefore, $ISCE(K_n) = (2n-2)^{\frac{3}{2}}$.

Theorem 4.2. *The inverse sum connectivity energy of the cycle graph C_{2n} is*

$$ISCE(C_{2n}) = 8 + \sum_{m=1, m\neq n}^{2n-1} |4\cos\frac{\pi m}{n}|.$$

Proof. The inverse sum connectivity matrix corresponding to the cycle graph C_{2n} is

$$ISC(C_{2n}) = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & \dots & 0 & 2 \\ 2 & 0 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & \dots & 2 & 0 \end{bmatrix}.$$

This is a circullant matrix of order $2n$. Its eigenvalues are

$$\lambda_m = \begin{cases} 4, & \text{for } m = 0 \\ -4, & \text{for } m = n \\ 4\cos\frac{\pi m}{n}, & \text{for } 0 < m < n,\ n < m \leq 2n-1. \end{cases}$$

Therefore the inverse sum connectivity energy is

$SDD(C_{2n}) = |-4| + |4| + \sum_{m=1, m\neq n}^{2n-1} |4\cos\frac{\pi m}{n}|$ and finally we get

$$ISCE(C_{2n}) = 8 + \sum_{m=1, m\neq n}^{2n-1} |4\cos\frac{\pi m}{n}|.$$

Theorem 4.3. *The inverse sum connectivity energy of the star graph $K_{1,n-1}$ is*

$$ISCE(K_{1,n-1}) = 2\sqrt{n(n-1)}.$$

Proof. Let $K_{1,n-1}$ be the star graph with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ with v_0 denotes the central vertex. The inverse sum connectivity matrix is

$$ISC(K_{1,n-1}) = \begin{bmatrix} 0 & \sqrt{n} & \sqrt{n} & \dots & \sqrt{n} & \sqrt{n} \\ \sqrt{n} & 0 & 0 & \dots & 0 & 0 \\ \sqrt{n} & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{n} & 0 & 0 & \dots & 0 & 0 \\ \sqrt{n} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

The characteristic equation becomes

$$\lambda^{n-2}(\lambda^2 - n(n-1)) = 0$$

and therefore, the spectrum would be

$$Spec_{ISC}(K_{1,n-1}) = \begin{pmatrix} -\sqrt{n(n-1)} & 0 & \sqrt{n(n-1)} \\ 1 & n-2 & 1 \end{pmatrix}.$$

Therefore,

$$ISCE(K_{1,n-1}) = 2\sqrt{n(n-1)}.$$

Definition 4.4. *The friendship graph, denoted by F_n^3, is defined as the graph obtained by taking n copies of the cycle graph C_3 with a vertex in common.*

It is clear that $|V(F_n^3)| = 2n + 1$.

Theorem 4.5. *The inverse sum connectivity energy of the friendship graph F_n^3 is*

$$ISCE(F_n^3) = 8n.$$

Proof. Let F_n^3 be the friendship graph with $2n+1$ vertices and let v_0 be the common vertex. The inverse sum connectivity matrix is

$$\begin{bmatrix} 0 & \sqrt{2n+2} & \sqrt{2n+2} & \sqrt{2n+2} & \sqrt{2n+2} & \dots & \sqrt{2n+2} & \sqrt{2n+2} \\ \sqrt{2n+2} & 0 & 2 & 0 & 0 & \dots & 0 & 0 \\ \sqrt{2n+2} & 2 & 0 & 0 & 0 & \dots & 0 & 0 \\ \sqrt{2n+2} & 0 & 0 & 0 & 2 & \dots & 0 & 0 \\ \sqrt{2n+2} & 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{2n+2} & 0 & 0 & 0 & 0 & \dots & 0 & 2 \\ \sqrt{2n+2} & 0 & 0 & 0 & 0 & \dots & 2 & 0 \end{bmatrix}.$$

The characteristic equation becomes

$$(\lambda-2)^{n-1}(\lambda+2)^n(\lambda+2n)(\lambda-(2n+2))=0$$

implying that the spectrum is

$$Spec_{SDD}(F_n^3)=\begin{pmatrix} -2 & 2 & -2n & 2n+2 \\ n & n-1 & 1 & 1 \end{pmatrix}.$$

Therefore, we get

$$ISCE(F_n^3)=8n.$$

Theorem 4.6. *The inverse sum connectivity energy of the cocktail party graph $K_{n\times 2}$ is*

$$ISCE(K_{n\times 2})=4(n-1)^2.$$

Proof. Let $K_{n\times 2}$ be the cocktail party graph of order $2n$ having vertex set $\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$. The inverse sum connectivity matrix is

$$ISC(K_{n\times 2})=\begin{bmatrix} 0 & 0 & \sqrt{4n-4} & \sqrt{4n-4} & \dots & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} \\ 0 & 0 & \sqrt{4n-4} & \sqrt{4n-4} & \dots & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} \\ \sqrt{4n-4} & \sqrt{4n-4} & 0 & 0 & \dots & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} \\ \sqrt{4n-4} & \sqrt{4n-4} & 0 & 0 & \dots & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \dots & 0 & 0 & \sqrt{4n-4} & \sqrt{4n-4} \\ \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \dots & 0 & 0 & \sqrt{4n-4} & \sqrt{4n-4} \\ \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \dots & \sqrt{4n-4} & \sqrt{4n-4} & 0 & 0 \\ \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \sqrt{4n-4} & \dots & \sqrt{4n-4} & \sqrt{4n-4} & 0 & 0 \end{bmatrix}.$$

In that case, the characteristic equation is

$$\lambda^n(\lambda+4)^{n-1}(\lambda-(4n-4))=0$$

and hence the spectrum becomes

$$Spec_{ISC}(K_{n\times 2}) = \begin{pmatrix} 2n-2 & 0 & 2(n-1)^2 \\ n-1 & n & 1 \end{pmatrix}.$$

Therefore we arrive at the required result:

$$ISCE(K_{n\times 2}) = 4(n-1)^2.$$

Theorem 4.7. *The inverse sum connectivity energy of the double star graph $S_{n,n}$ is*

$$ISCE(S_{n,n}) = 2\sqrt{4n^2+2n-4}.$$

Proof. The inverse sum connectivity matrix is

$$ISC(S_{n,n}) = \begin{bmatrix} 0 & \sqrt{n+1} & \sqrt{n+1} & \dots & \sqrt{n+1} & \sqrt{2n} & 0 & 0 & \dots & 0 \\ \sqrt{n+1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \sqrt{n+1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{n+1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \sqrt{2n} & 0 & 0 & \dots & 0 & 0 & \sqrt{n+1} & \sqrt{n+1} & \dots & \sqrt{n+1} \\ 0 & 0 & 0 & \dots & 0 & \sqrt{n+1} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \sqrt{n+1} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \sqrt{n+1} & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then the characteristic equation becomes

$$\lambda^{2n-4}(\lambda^2+\sqrt{2n}\lambda-(n^2-1))(\lambda^2-\sqrt{2n}\lambda-(n^2-1))=0.$$

Hence, the spectrum would be

$$Spec_{ISC}(S_{n,n}) = \begin{pmatrix} 0 & \frac{\sqrt{2n}+\sqrt{4n^2+2n-4}}{2} & \frac{\sqrt{2n}+\sqrt{4n^2+2n-4}}{2} & \frac{\sqrt{2n}+\sqrt{4n^2+2n-4}}{2} & \frac{\sqrt{2n}+\sqrt{4n^2+2n-4}}{2} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1+\sqrt{1+\frac{(n-1)(n^2+1)^2}{n^2}} & 1-\sqrt{1+\frac{(n-1)(n^2+1)^2}{n^2}} \\ 1 & 1 \end{pmatrix}$$

and therefore, we get

$$ISCE(S_{n,n}) = 2\sqrt{4n^2+2n-4}.$$

Definition 4.8. *A graph obtained by joining n copies of the cycle graph C_4 of length 4 at a common vertex is called a Dutch windmill graph. It will be denoted by D_4^n.*

It is clear that the Dutch windmill graph D_4^n has $3n+1$ vertices and $4n$ edges.

Theorem 4.9. *The inverse sum connectivity energy of the Dutch windmill graph D_4^n is*

$$ISCE(D_4^n) = 2\sqrt{8}(n-1) + 4\sqrt{n^2+n+2}.$$

Proof. Recall that D_4^n has $3n+1$ vertices. Then the inverse sum connectivity matrix is

$$ISC(D_4^n) = \begin{bmatrix} 0 & \sqrt{2n+2} & 0 & \sqrt{2n+2} & \sqrt{2n+2} & \dots & 0 & \sqrt{2n+2} \\ \sqrt{2n+2} & 0 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & \dots & 0 & 0 \\ \sqrt{2n+2} & 0 & 0 & 0 & 2 & \dots & 0 & 0 \\ \sqrt{2n+2} & 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2 \\ \sqrt{2n+2} & 0 & 0 & 0 & 0 & \dots & 2 & 0 \end{bmatrix}.$$

Hence the characteristic equation will be

$$(\lambda - \sqrt{8})^{n-1}(\lambda + \sqrt{8})^{n-1}\lambda^{n+1}(\lambda^2 - 4(n^2+n+2)) = 0$$

and therefore the spectrum becomes

$$Spec_{ISC}(D_4^n) = \begin{pmatrix} \sqrt{8} & -\sqrt{8} & 0 & 2\sqrt{n^2+n+2} & -2\sqrt{n^2+n+2} \\ n-1 & n-1 & n+1 & 1 & 1 \end{pmatrix}.$$

Therefore, it is directly seen that

$$ISCE(D_4^n) = 2\sqrt{8}(n-1) + 4\sqrt{n^2+n+2}.$$

Theorem 4.10. *The inverse sum connectivity energy of crown graph S_n^0 is*

$$ISCE(S_n^0) = 2(2n-2)^{\frac{3}{2}}.$$

Proof. Let S_n^0 be the crown graph of order $2n$ and let the vertex set of this graph

be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The inverse sum connectivity matrix of S_n^0 is

$$ISC(S_n^0) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \sqrt{2n-2} & \dots & \sqrt{2n-2} & \sqrt{2n-2} \\ 0 & 0 & 0 & \dots & 0 & \sqrt{2n-2} & 0 & \dots & \sqrt{2n-2} & \sqrt{2n-2} \\ 0 & 0 & 0 & \dots & 0 & \sqrt{2n-2} & \sqrt{2n-2} & \dots & 0 & \sqrt{2n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \sqrt{2n-2} & \sqrt{2n-2} & \dots & \sqrt{2n-2} & 0 \\ 0 & \sqrt{2n-2} & \sqrt{2n-2} & \dots & \sqrt{2n-2} & 0 & 0 & \dots & 0 & 0 \\ \sqrt{2n-2} & 0 & \sqrt{2n-2} & \dots & \sqrt{2n-2} & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{2n-2} & \sqrt{2n-2} & 0 & \dots & \sqrt{2n-2} & 0 & 0 & \dots & 0 & 0 \\ \sqrt{2n-2} & \sqrt{2n-2} & \sqrt{2n-2} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Therefore the characteristic equation is

$$(\lambda-2)^{n-1}(\lambda+2)^{n-1}(\lambda+(2n-2))(\lambda-(2n-2)) = 0$$

implying that the spectrum is

$$Spec_{SDD}(S_n^0) = \begin{pmatrix} 2n-2 & -(2n-2) & 2 & -2 \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Therefore we obtain

$$ISCE(S_n^0) = 2(2n-2)^{\frac{3}{2}}.$$

Theorem 4.11. *The inverse sum connectivity energy of the complete bipartite graph $K_{m,n}$ of order $m \times n$ with vertex set $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ is*

$$ISCE(K_{m,n}) = 2\sqrt{mn(m+n)}.$$

Proof. The inverse sum connectivity matrix of the complete bipartite graph $K_{m,n}$ is

$$ISC(K_{m,n}) = \begin{bmatrix} 0 & 0 & 0 & \dots & \sqrt{m+n} & \sqrt{m+n} & \sqrt{m+n} \\ 0 & 0 & 0 & \dots & \sqrt{m+n} & \sqrt{m+n} & \sqrt{m+n} \\ 0 & 0 & 0 & \dots & \sqrt{m+n} & \sqrt{m+n} & \sqrt{m+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sqrt{m+n} & \sqrt{m+n} & \sqrt{m+n} & \dots & 0 & 0 & 0 \\ \sqrt{m+n} & \sqrt{m+n} & \sqrt{m+n} & \dots & 0 & 0 & 0 \\ \sqrt{m+n} & \sqrt{m+n} & \sqrt{m+n} & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Then the characteristic equation is

$$\lambda^{m+n-2}(\lambda - \sqrt{mn(m+n)})(\lambda + \sqrt{mn(m+n)}) = 0$$

and therefore the spectrum is

$$Spec_{ISC}(K_{m,n}) = \begin{pmatrix} \sqrt{mn(m+n)} & 0 & -\sqrt{mn(m+n)} \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

Therefore, we get

$$ISCE(K_{m,n}) = 2\sqrt{mn(m+n)}.$$

5. Inverse sum connectivity energy of complements

Definition 5.1. *[6] Let G be a graph and $P_k = \{V_1, V_2, \cdots, V_k\}$ be a partition of its vertex set V. Then the k-complement of G is denoted by $\overline{(G)_k}$ and obtained as follows: For all V_i and V_j in P_k, $i \neq j$, remove the edges between V_i and V_j and add the edges between the vertices of V_i and V_j which are not in G.*

Definition 5.2. *[6] Let G be a graph and $P_k = \{V_1, V_2, \cdots, V_k\}$ be a partition of its vertex set V. Then the $k(i)$-complement of G is denoted by $\overline{(G)_{k(i)}}$ and obtained as follows: For each set V_r in P_k, remove the edges of G joining the vertices within V_r and add the edges of $\overline{G}$ (complement of G) joining the vertices of V_r.*

There is usually a nice relation between some properties of a graph and its complement. Here we investigate the relation between some special graph classes and their complements in terms of the inverse sum connectivity energy.

Theorem 5.3. *The inverse sum connectivity energy of the complement $\overline{K_n}$ of the complete graph K_n is*

$$SDDE(\overline{K_n}) = 0.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The inverse sum connectivity connectivity matrix of the complement of the complete graph K_n is

$$SDD(\overline{K_n}) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Clearly, the characteristic equation is $\lambda^n = 0$ implying

$$E(\overline{K_n}) = 0.$$

Theorem 5.4. *The inverse sum connectivity energy of the complement $\overline{K_{1,n-1}}$ of the star graph $K_{1,n-1}$ is*

$$ISCE(\overline{K_{1,n-1}}) = 2(n-2)\sqrt{2n-4}.$$

Proof. Let $\overline{K_{1,n-1}}$ be the complement of the star graph with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ where v_0 is the central. The inverse sum connectivity matrix is

$$ISC(\overline{K_{1,n-1}}) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \sqrt{2n-4} & \dots & 2 & \sqrt{2n-4} \\ 0 & \sqrt{2n-4} & 0 & \dots & \sqrt{2n-4} & \sqrt{2n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \sqrt{2n-4} & \sqrt{2n-4} & \dots & 0 & \sqrt{2n-4} \\ 0 & \sqrt{2n-4} & \sqrt{2n-4} & \dots & \sqrt{2n-4} & 0 \end{bmatrix}.$$

The corresponding characteristic equation is

$$\lambda(\lambda - (2n-4))(\lambda+2)^{n-2} = 0$$

and therefore the spectrum is

$$Spec_{ISC}(\overline{K_{1,n-1}}) = \begin{pmatrix} -\sqrt{2n-4} & 0 & n-2\sqrt{2n-4} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Therefore,

$$ISCE(\overline{K_{1,n-1}}) = 2(n-2)\sqrt{2n-4}.$$

Theorem 5.5. *The inverse sum connectivity energy of the complement* $\overline{K_{n\times 2}}$ *of the cocktail party graph* $K_{n\times 2}$ *of order* $2n$ *is*

$$ISCE\overline{(K_{n\times 2})} = 2n\sqrt{2}.$$

Proof. Let $K_{n\times 2}$ be the cocktail party graph of order $2n$ having the vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The corresponding inverse sum connectivity matrix is

$$ISC(\overline{K_{n\times 2}}) = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \sqrt{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \sqrt{2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the characteristic equation becomes

$$(\lambda+\sqrt{2})^n(\lambda-\sqrt{2})^n=0$$

implying that the spectrum would be

$$Spec_{ISC}(\overline{K_{n\times 2}})=\begin{pmatrix}\sqrt{2} & -\sqrt{2}\\ n & n\end{pmatrix}.$$

Therefore,

$$ISCE(\overline{K_{n\times 2}})=2\sqrt{2}n.$$

Theorem 5.6. *The inverse sum connectivity energy of* 2*-complement of cocktail party graph* $K_{n\times 2}$ *is*

$$ISCE(\overline{K_{n\times 2_{(2)}}})=(2n-2)\sqrt{8n}.$$

Proof. Consider the 2-complement $\overline{K_{n\times 2_{(2)}}}$ of the cocktail party graph $K_{n\times 2_{(2)}}$. The inverse sum connectivity matrix is

$$ISC(\overline{K_{n\times 2_{(2)}}})=\begin{bmatrix}
0 & \sqrt{2n} & \sqrt{2n} & \dots & \sqrt{2n} & \sqrt{2n} & 0 & \dots & 0 & 0\\
\sqrt{2n} & 0 & \sqrt{2n} & \dots & \sqrt{2n} & 0 & \sqrt{2n} & \dots & 0 & 0\\
\sqrt{2n} & \sqrt{2n} & 0 & \dots & \sqrt{2n} & 0 & 0 & \dots & \sqrt{2n} & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\sqrt{2n} & \sqrt{2n} & \sqrt{2n} & \dots & 0 & 0 & 0 & \dots & 0 & \sqrt{2n}\\
\sqrt{2n} & 0 & 0 & \dots & 0 & 0 & \sqrt{2n} & \dots & \sqrt{2n} & \sqrt{2n}\\
0 & \sqrt{2n} & 0 & \dots & 0 & \sqrt{2n} & 0 & \dots & \sqrt{2n} & \sqrt{2n}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \sqrt{2n} & \dots & 0 & \sqrt{2n} & \sqrt{2n} & \dots & 0 & \sqrt{2n}\\
0 & 0 & 0 & \dots & \sqrt{2n} & \sqrt{2n} & \sqrt{2n} & \dots & \sqrt{2n} & 0
\end{bmatrix}.$$

The characteristic polynomial is

$$\lambda^{n-1}(\lambda+4)^{n-1}(\lambda-(2n-4))=0$$

and therefore, the inverse sum connectivity spectra is

$$Spec(\overline{(K_{n\times 2})_{(2)}})=\begin{pmatrix}\sqrt{8n} & 0 & 2n-4 & 2n\\ n-1 & n-1 & 1 & 1\end{pmatrix}.$$

implying that the inverse sum connectivity energy is

$$ISCE(\overline{(K_{n\times 2})_{(2)}})=(2n-2)\sqrt{8n}.$$

6. Inverse sum connectivity energy of graphs with one edge deleted

Theorem 6.1. *Let e be an edge of the complete graph K_n. Then*

$$ISC(K_n - e) = (n-3)\sqrt{2n-2} + \sqrt{2n^3 + 2n^2 - 26n + 30}.$$

Proof. The inverse sum connectivity matrix for $K_n - e$ is

$$ISC(K_n - e) = \begin{bmatrix} 0 & 0 & \sqrt{2n-3} & \dots & \sqrt{2n-3} & \sqrt{2n-3} \\ 0 & 0 & \sqrt{2n-3} & \dots & \sqrt{2n-3} & \sqrt{2n-3} \\ \sqrt{2n-3} & \sqrt{2n-3} & 0 & \dots & \sqrt{2n-2} & \sqrt{2n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{2n-3} & \sqrt{2n-3} & \sqrt{2n-2} & \dots & 0 & \sqrt{2n-2} \\ \sqrt{2n-3} & \sqrt{2n-3} & \sqrt{2n-2} & \dots & \sqrt{2n-2} & 0 \end{bmatrix}.$$

Therefore the characteristic equation is

$$\lambda(\lambda + \sqrt{2n+2})^{n-3}\left(\lambda^2 - (n-3)\sqrt{2n-2}\lambda - (4n^2 - 14n + 12)\right) = 0$$

implying that the spectrum would be
$Spec_{ISC}(K_n - e) =$
$$\begin{pmatrix} -\sqrt{2n+2} & 0 & \frac{n-3+\sqrt{2n-2}+(\sqrt{2n^3+2n^2-26n+30})}{2} & \frac{n-3+\sqrt{2n-2}-(\sqrt{2n^3+2n^2-26n+30})}{2} \\ n-3 & 1 & 1 & 1 \end{pmatrix}.$$
Therefore, $ISCE(K_n - e) = (n-3)\sqrt{2n-2} + \sqrt{2n^3 + 2n^2 - 26n + 30}$

Theorem 6.2. *Let e be an edge of the complete bipartite graph $K_{n,n}$. The inverse sum connectivity energy of $K_{n,n} - e$ is*

$$ISCE(K_{n,n} - e) = 2\sqrt{2n^3 + 4n^2 - 10n + 4}.$$

Proof. The inverse sum connectivity matrix for $K_{n,n} - e$ is

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \sqrt{2n-1} & \sqrt{2n-1} \\ 0 & 0 & 0 & \dots & \sqrt{2n-1} & \sqrt{2n} & \sqrt{2n} \\ 0 & 0 & 0 & \dots & \sqrt{2n-1} & \sqrt{2n} & \sqrt{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \sqrt{2n-1} & \sqrt{2n-1} & \dots & 0 & 0 & 0 \\ \sqrt{2n-1} & \sqrt{2n} & \sqrt{2n} & \dots & 0 & 0 & 0 \\ \sqrt{2n-1} & \sqrt{2n} & \sqrt{2n} & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Then the characteristic equation is

$$\lambda^{2n-4}(\lambda^2 + (n-1)\sqrt{2n}\lambda - (2n^2 - 3n + 1))(\lambda^2 - (n-1)\sqrt{2n}\lambda - (2n^2 - 3n + 1)) = 0$$

and hence, the spectrum would be

$$Spec_{ISC}(K_{n,n}-e)=\begin{pmatrix} \frac{(n-1)\sqrt{2n}+A}{2} & \frac{(n-1)\sqrt{2n}-A}{2} & \frac{-(n-1)\sqrt{2n}+A}{2} & \frac{-(n-1)\sqrt{2n}-A}{2} & 0 \\ 1 & 1 & 1 & 1 & 2n-4 \end{pmatrix}$$

where $A=\sqrt{2n^3+4n^2-10n+4}$.
Therefore,

$$ISCE(K_{n,n}-e)=2\sqrt{2n^3+4n^2-10n+4}.$$

The following result can easily be proven as above:

Lemma 6.3. *Let $K_{1,n-1}$ be the star graph with n vertices and let e be an edge of it. Then $ISCE(K_{1,n-1}-e)=ISCE(K_{1,n-2})$ for $n\geq 3$.*

7. Some open problems

Open problem 7.1. *With respect to inverse sum connectivity, determine the class of graphs which are co-spectral and characterize them.*

Open problem 7.2. *With respect to inverse sum connectivity, determine the class of graphs which are hyperenergetic and characterize them.*

Open problem 7.3. *With respect to inverse sum connectivity, determine the class of graphs whose inverse sum connectivity energy and inverse sum connectivity energy of their complements are equal.*

Open problem 7.4. *With respect to inverse sum connectivity, determine the class of non-co-spectral graphs which are equienergetic.*

Open problem 7.5. *Determine the class of graphs whose inverse sum connectivity energy is equal to usual energy.*

References

[1] V. Alexander, Upper and lower bounds of symmetric division deg index, *Iranian Journal of Mathematical Chemistry*, 5 (2) (2014), 91-98.

[2] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz*, 103 (1978), 1-22.

[3] I. Gutman, The energy of a graph: old and new results, *Combinatorics and applications, A. Betten, A. Khoner, R. Laue and A. Wassermann, (Eds.), Springer, Berlin*, (2001), 196-211.

[4] K. N Prakasha, The inverse sum connectivity index of graphs (Communicated).

[5] M. Randić, On characterization of molecular branching, *J. Am. Chem. Soc.*, 97 (1975), 6609-6615.

[6] E. Sampathkumar, L. Pushpalatha, C. V. Venkatachalam and Pradeep Bhat, Generalized complements of a graph, *Indian J. Pure Appl. Math.*, 29(6) (1998), 625-639.

[7] R. Todeschini, V. Consonni, Handbook of Molecular Descriptors, *Wiley-VCH, Weinheim*, (2000), 84-90.

[8] R. Todeschini, V. Consonni, Molecular Descriptors for Chemoinformatics, *Wiley-VCH, Weinheim*, (2009), 161-172.

[9] B. Zhou, N. Trinajstic, On Sum-Connectivity Matrix and Sum-Connectivity Energy of (Molecular) Graphs, *Acta Chim. Slov*, 57 (2010), 518-523.

[10] B. Zhou, N. Trinajstic, On a novel connectivity index, *J. Math. Chem.*, 46 (2009), 1252-1270.