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ON MODULAR RELATIONS FOR THE FUNCTIONS ANALOGOUS TO ROGER'S-RAMANUJAN FUNCTIONS WITH APPLICATIONS TO PARTITIONS

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Dedicated to Professor G. E. Andrews, on the occasion of his Seventieth birthday

Abstract: In this paper, we establish modular relations involving the functions, $S(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n}$ and $T(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}$, which are analogous to Ramanujan's modular identities. Furthermore, we extract some partition results from them.

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1. Introduction

The famous Roger's-Ramanujan functions $G(q)$ and $H(q)$ are

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{1.1}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.2}$$

where, as customary

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$

and

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1$$

Birch [6] published 40 identities conjectured by Ramanujan involving the functions $G(q)$ and $H(q)$. Rogers [14], Watson [15] had proved 16 of these identities. By combining and extending the methods of Rogers and Watson, in his Ph. D. thesis, Bressoud [7] proved fifteen, out of forty identities. His published paper [8] contains proofs of the general identities from [7], that he developed in order to prove Ramanujan's identities. Biagioli [5] used modular forms to prove eight more identities of Ramanujan. On employing Ramanujan's theta function identities and modular equations found in [13], Berndt, *et al.* [3], Berndt and Yesilyurt [4] found new proofs of all forty identities in the spirit of Ramanujan.

Two identities analogous to the Rogers-Ramanujan identities are the so called Göllnitz-Gordon identities [10,11] which are expressed by the equations

$$L(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}, \tag{1.3}$$

and

$$M(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{1.4}$$

The Rogers-Ramanujan and the Göllnitz-Gordon functions share some remarkable properties. For example, the quotients of $G(q)$ and $H(q)$ gives the celebrated Rogers-Ramanujan continued fraction, while the quotient of $L(q)$ and $M(q)$ gives the Ramanujan Göllnitz-Gordon continued fraction [12, p.229], [10,11]. Moreover (1.1),(1.2),(1.3),(1.4) all have elegant partition interpretations. For details see [1]. By using methods of Rogers [14] and Bressoud [7], Huang [12], obtained eighteen modular relations involving $L(q)$ and $M(q)$, which are analogous to the forty identities of Ramanujan for $G(q)$ and $H(q)$. Huang [12] also found applications of these modular relations to partitions and colored partitions.

Motivated by these, in Section 2, we establish certain modular relations for the functions $S(q)$ and $T(q)$, which are defined as

$$S(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} = \frac{1}{(q^2;q^4)_\infty}, \tag{1.5}$$

and

$$T(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} = \frac{(q^2;q^4)_\infty}{(q;q^4)_\infty (q^3;q^4)_\infty}. \tag{1.6}$$

The quotient of $S(q)$ and $T(q)$ gives the Ramanujan continued fraction [2, p.221, Entry 1(i)].

The relations that we obtain are rich with applications to colored partitions. In Section 3, we derive some interesting results of colored partitions. We now recall some definitions and certain identities stated by Ramanujan which will be used in the next sections. The Ramanujan's definition of general theta function is given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a,ab)_\infty(-b,ab)_\infty(ab,ab)_\infty, \qquad |ab|<1. \tag{1.7}$$

The three most important special cases of (1.7) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty^2,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

We also define

$$\chi(q) = (-q;q^2)_\infty.$$

The modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;(1-\alpha)\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;(1-\beta)\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^n,$$

with

$$(a)_k = a(a+1)(a+2)...(a+k-1).$$

2. Identities for the Functions $S(q)$ and $T(q)$

Let

$$f_n := f(-q^n),$$

Then from Entries 24 and 25 [2, pp.39-40], it is easy to see that

$$\varphi(q)=\frac{f_2^5}{f_1^2f_4^2},\quad \varphi(-q)=\frac{f_1^2}{f_2},\quad \psi(q)=\frac{f_2^2}{f_1},$$

$$\psi(-q) = \frac{f_1f_4}{f_2}, \text{ and } \chi(q) = \frac{f_2^2}{f_1f_4}. \tag{2.1}$$

Theorem 2.1. We have

$$\text{(i)} \qquad T^2(q^4)+2qS^2(q^4)=\frac{f_2^5}{f_1^2f_4^2f_8}, \tag{2.2}$$

and

$$\text{(ii)} \qquad T^2(q^4)-2qS^2(q^4)=\frac{f_1^2}{f_2f_8}. \tag{2.3}$$

Proof. From Entry 25(i) and (ii) of Chapter 16 of Ramanujan's second notebook [13] [2,p. 40], we have

$$\varphi(q)+\varphi(-q)=2\varphi(q^4), \tag{2.4}$$

and

$$\varphi(q)-\varphi(-q)=4q\psi(q^8). \tag{2.5}$$

Adding (2.4) and (2.5), and then employing (2.1) in the resulting identity and after dividing throughout by f_8, we obtain (2.2). Subtracting (2.5) from (2.4) and then employing (2.1) in the resulting identity and after dividing throughout by f_8, we obtain (2.3).

Theorem 2.2. We have

$$T^2(q)T^2(q^3)-4qS^2(q)S^2(q^3)=\frac{f_1^2f_3^2}{f_2^2f_6^2}. \tag{2.6}$$

Proof. From Entry 3(i) and (ii) of Chapter 19 of [2, p.223], we have

$$q\psi(q^2)\psi(q^6)=\sum_{n=0}^{\infty}\frac{q^{6n+1}}{1-q^{12n+2}}-\sum_{n=0}^{\infty}\frac{q^{6n+5}}{1-q^{12n+10}}$$

and

$$\varphi(q)\varphi(q^3)=1+2\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{q^k}{1+(-q)^k},$$

where $\left(\begin{array}{c} k \\ 3 \end{array}\right)$ the Legendre's symbol. From the above two identities, one can easily show that

$$4q\psi(q^2)\psi(q^6) = \varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3).$$

Now employing (2.1) in the above identity and then dividing throughout by $f_2 f_6$, we obtain (2.6).

Theorem 2.3. We have

$$S^4(q)T^4(q^3) - qT^4(q)S^4(q^3) \ = \ \frac{f_1 f_3}{f_2 f_6}. \tag{2.7}$$

Proof. Let β be of degree 3 over α. Then from Entry 5(xiii) of Chapter 19 [2, p. 231], we have

$$(\alpha)^{1/2} - (\beta)^{1/2} = 2(\alpha\beta)^{1/8}[1 - (\alpha\beta)^{1/4}].$$

Employing Entry 5(ii) of Chapter 19 [2, p. 230] in the right side of the above identity, we deduce

$$(\alpha)^{1/2} - (\beta)^{1/2} = 2(\alpha\beta)^{1/8}[(1-\alpha)(1-\beta)]^{1/4}.$$

Now using Entry 10 (i),(ii) and Entry 11 (i) and (ii) of Chapter 17 of [2, pp.122-123], in the above identity, we deduce

$$\frac{\psi^4(q^2)}{\psi^4(q)} - q\frac{\psi^4(q^6)}{\psi^4(q^3)} \ = \ 2\frac{\varphi(-q)\varphi(-q^3)\psi(q^2)\psi(q^6)}{\varphi(q)\varphi(q^3)\psi(q)\psi(q^3)}. \tag{2.8}$$

Using (1.5) and (1.6) in the left side of (2.7), we see that

$$\begin{aligned} S^4(q)T^4(q^3) - qT^4(q)S^4(q^3) & \\ &= \frac{f_4^4 f_6^8}{f_2^4 f_3^4 f_{12}^4} - q\frac{f_2^8 f_{12}^4}{f_1^4 f_4^4 f_6^4} \\ &= \frac{f_2^8 f_6^8}{(f_1 f_3 f_4 f_{12})^4}\left[\frac{\psi^4(q^2)}{\psi^4(q)} - q\frac{\psi^4(q^6)}{\psi^4(q^3)}\right] \\ &= \frac{f_1 f_3}{f_2 f_6}. \end{aligned}$$

where, we have used (2.8). This completes the proof.

Theorem 2.4. We have

$$S^2(q)T^2(q^5) - qS^2(q^5)T^2(q) = \frac{f_1f_5}{f_2f_{10}}. \tag{2.9}$$

Proof. From Entry 10(v) of Chapter 19 of Ramanujan notebook [2, p. 262], we have

$$\psi^2(q) - q\psi^2(q^5) = f_1f_5 \tag{2.10}$$

From [2. p. 276, eq.(12.32)], we have

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)} - q\frac{\psi^2(-q^5)}{\psi^2(-q)} = 1 - q\frac{\psi^2(q^5)}{\psi^2(q)}.$$

Using (2.10) in the right side of the above identity, we obtain

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)} - q\frac{\psi^2(-q^5)}{\psi^2(-q)} = \frac{f_1f_5^3}{f_2^3f_{10}}.$$

Multiplying throughout by $\frac{f_2^2f_{10}^2}{\varphi^2(-q^{10})\psi^2(-q^5)}$, we obtain

$$\frac{f_2^2f_{10}^2}{\varphi^2(-q^2)\psi^2(-q^5)} - q\frac{f_{10}^2f_2^2}{\varphi^2(-q^{10})\psi^2(-q)} = \frac{f_1f_5}{f_2f_{10}}.$$

Now using easily deducible identity $T(q) = \frac{f_2}{\varphi(-q)}$ and $S(q) = \frac{f_2}{\varphi(-q^2)}$ in the above equation, we obtain the required result.

Theorem 2.5. We have

$$T(q)T(q^7) - 2qS(q)S(q^7) = 2\frac{f_1^2f_4^2f_7^2f_{28}^2}{f_2^4f_{14}^4}. \tag{2.11}$$

Proof. From Entry 19 (i) of Chapter 19 [2, p. 314], we have

$$1 - (\alpha\beta)^{1/8} = [(1-\alpha)(1-\beta)]^{1/8}, \tag{2.12}$$

where β has degree 7 over α. Also, from Entry 10 (iii) and Entry (ii) of Chapter 17 [2, pp. 122-123], we deduce that

$$\sqrt{2}q^{1/8}\frac{\psi(-q)}{\varphi(-q^2)} = (\alpha)^{1/8}.$$

Hence, if β has degree 7 over α, then

$$2q\frac{\psi(-q)\psi(-q^7)}{\varphi(-q^2)\varphi(-q^{14})} = (\alpha\beta)^{1/8}.$$

From Entry 10 (i) and (iii) of the Chapter 17 [2, p. 122], it follows that

$$\frac{\varphi(-q)\varphi(-q^7)}{\varphi(-q^2)\varphi(-q^{14})} = [(1-\alpha)(1-\beta)]^{1/8}.$$

From (2.12) and above two identities, we deduce that

$$1 - 2q\frac{\psi(-q)\psi(-q^7)}{\varphi(-q^2)\varphi(-q^{14})} = \frac{\varphi(-q)\varphi(-q^7)}{\varphi(-q^2)\varphi(-q^{14})},$$

which can be rewritten as

$$\frac{f_2 f_{14}}{\psi(-q)\psi(-q^7)} - 2q\frac{f_2 f_{14}}{\varphi(-q^2)\varphi(-q^{14})} = \frac{f_1^2 f_4^2 f_7^2 f_{28}^2}{f_2^4 f_{14}^4}.$$

Now using the fact that $T(q) = \frac{f_2}{\psi(-q)}$ and $S(q) = \frac{f_2}{\varphi(-q^2)}$, we obtain the required result.

Theorem 2.6. We have

$$\text{(i)} \qquad S(q)T(q^9) - qT(q)S(q^9) = \frac{f_1^3 f_6 f_9^3}{f_2^3 f_3 f_{18}^3},$$

and

$$\text{(ii)} \qquad S(q)T^2(q) + S(q^2)T^2(q^9) = 2\frac{f_1 f_{12}^3 f_{36}}{f_6 f_9 f_{18}}.$$

Proof. Part (i) follows from the following identities found in Entry 4 (ii) [2, p. 358] and Entry 2(ii) [2, p. 349] of Chapter 20, respectively:

$$\frac{\varphi(-q^2)}{\varphi(-q^{18})} + \frac{\psi(q)}{q\psi(q^9)} = 3 + \frac{\psi(-q)}{q\psi(-q^9)},$$

and

$$\psi(q) - 3q\psi(q^9) = \frac{\varphi(-q)}{\chi(-q^3)}.$$

Part (ii) follows from the following identities found in Entry 4(i) [2, p. 358] and Entry 1(ii) [2, p. 345] of Chapter 20 [2], respectively:

$$\frac{\varphi(-q^{18})}{\varphi(-q^2)} + q\left(\frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)}\right) = 1,$$

and

$$2q\frac{\chi(-q^3)}{\chi^3(-q^9)} = 1 - \frac{\varphi(-q)}{\varphi(-q^9)}.$$

Theorem 2.7. We have

(i) $$T(q)T(q^{15})+2q^2S(q)S(q^{15}) = \frac{f_6^5 f_{10}^5}{f_1 f_3^2 f_5^2 f_{15} f_{12}^2 f_{20}^2},$$

and

(ii) $$T(q^3)T(q^5)+2qS(q^3)S(q^5) = \frac{f_6^5 f_{10}^5}{f_3^3 f_5^3 f_{12}^2 f_{20}^2}.$$

Proof. Part (i) follows from the following identity found in Entry 9 (iii) of Chapter 20 [2, p. 377]:

$$\varphi(-q^2)\varphi(-q^{30}) + 2q^2\psi(q)\psi(q^{15}) = \varphi(q^3)\varphi(q^5).$$

Part (ii) follows from the following identity found in Entry 9 (ii) of Chapter 20 [2, p. 377]:

$$\varphi(-q^6)\varphi(-q^{10}) + 2q\psi(q^3)\psi(q^5) = \varphi(q)\varphi(q^{15}).$$

3. Application to the Theory of Partition

The identities obtained in Section 2 have applications to the theory of partitions. In this section, we present partition interpretations of some of the results obtained in previous section. First, we recall definitions of a colored partitions and the required generating function.

Definition 3.1. Partitions of a positive integer into parts with colors are colored partitions.

For example, if 1 is allowed to have 2 colors, then all the (colored) partitions of 2 are 2, $1_r + 1_r, 1_g + 1_g$ and $1_r + 1_g$, where we use the indices r (red) and g (green) to distinguish the two colors of 1. An important fact is that

$$\frac{1}{(q^a;q^b)_\infty^k}$$

is the generating function for the number of partitions of n where all the parts are congruent to $a(mod\ b)$ and have k colors.

For simplicity, in this section, we adopt the following standard notations:

$$(a_1, a_2, ..., a_n; q)_\infty := \prod_{j=1}^{\infty}(a_j;q)_\infty$$

and

$$(q^{r\pm};q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty$$

where r and s are positive integers with $r < s$.

Theorem 3.1. Let $P_1(n)$ denote the number of partitions of n into parts congruent to $\pm 2, \pm 4, \pm 6, (mod 16)$, with three colors. Let $P_2(n)$ denote the number of partitions of n into parts congruent to $\pm 2, \pm 4, \pm 6, 8 (mod 16)$ and parts congruent to $\pm 2, \pm 6, (mod 16)$ with three colors, parts congruent to $8 (mod 16)$ with four colors. And let $P_3(n)$ denote the number of partition of n into parts congruent to $\pm 1, \pm 3, \pm 5, \pm 7, 8 (mod 16)$ with two colors. Then

$$P_1(n) + 2P_2(n-1) = P_3(n).$$

Proof. By the definitions of $S(q)$ and $T(q)$, (2.2) is equivalent to

$$\begin{aligned} &\frac{1}{(q^{2\pm}, q^{2\pm}, q^{2\pm}, q^{4\pm}, q^{4\pm}, q^{4\pm}, q^{6\pm}, q^{6\pm}, q^{6\pm}; q^{16})_\infty} \\ &+2q\frac{1}{(q^{2\pm}, q^{2\pm}, q^{2\pm}, q^{4\pm}, q^{6\pm}, q^{6\pm}, q^{6\pm}, q^{8}, q^{8}, q^{8}, q^{8}; q^{16})_\infty} \\ &= \frac{1}{(q^{1\pm}, q^{1\pm}, q^{3\pm}, q^{3\pm}, q^{5\pm}, q^{5\pm}, q^{7\pm}, q^{7\pm}, q^{8}, q^{8}; q^{16})_\infty}. \end{aligned}$$

One can easily see that the three quotients of (3.1) represents the generating function for $P_1(n)$, $P_2(n)$ and $P_3(n)$ respectively. Hence (3.1) is equivalent to

$$\sum_{n=0}^{\infty} P_1(n)q^n + 2q\sum_{n=0}^{\infty} P_2(n)q^n = \sum_{n=0}^{\infty} P_3(n)q^n.$$

where we set $P_1(0) = P_2(0) = P_3(0) = 1$. Equating the coefficients on both sides yields the desired result.

Example 3.1. The following table verifies the case $n = 4$ in Theorem 3.1.

$P_1(4) = 9$	$P_2(3) = 0$	$P_3(4) = 9$
$2_r + 2_r$		$1_r + 1_r + 1_r + 1_r$
$2_r + 2_w$		$1_r + 1_r + 1_r + 1_w$
$2_r + 2_g$		$1_r + 1_r + 1_w + 1_w$
$2_w + 2_w$		$1_r + 1_w + 1_w + 1_w$
$2_w + 2_g$		$1_w + 1_w + 1_w + 1_w$
$2_g + 2_g$		$3_r + 1_r$
4_r		$3_r + 1_w$
4_w		$3_w + 1_r$
4_g		$3_w + 1_w$

Theorem 3.2. Let $P_1(n)$ denote the number of partitions of n into parts not congruent to $8(mod 16)$, and parts congruent to $\pm 1, \pm 3, \pm 5, \pm 7 (mod 16)$ with 2 colors, parts congruent to $\pm 4 (mod 16)$ with 3 colors.

Let $P_2(n)$ denote the number of partitions of n and parts congruent to $\pm 1, \pm 3, \pm 5, \pm 7 (mod 16)$ with two colors, and the parts congruent to $8(mod 16)$ with three colors.

Let $P_3(n)$ denote the number of partitions of n into parts congruent to $8(mod 16)$ with two colors. Then

$$P_1(n) - 2P_2(n-1) = P_3(n).$$

Proof. The identity (2.3) can be rewritten as

$$\frac{(q^8;q^{16})^2}{(q^4,q^{12};q^{16})^2_\infty} - \frac{2q}{(q^8;q^{16})^2_\infty}$$
$$= (q;q^2)^2_\infty (q^2,q^4,q^6;q^8)_\infty,$$

which is equivalent to

$$\frac{1}{(q^{1\pm},q^{1\pm},q^{2\pm},q^{3\pm},q^{3\pm},q^{4\pm},q^{4\pm},q^{4\pm},q^{5\pm},q^{5\pm},q^{6\pm},q^{7\pm},q^{7\pm},q^{16})_\infty}$$
$$-\frac{2q}{(q^{1\pm},q^{1\pm},q^{2\pm},q^{3\pm},q^{3\pm},q^{4\pm},q^{5\pm},q^{5\pm},q^{6\pm},q^{7\pm},q^{7\pm},q^8,q^8,q^8;q^{16})_\infty}$$
$$= \frac{1}{(q^8,q^8;q^{16})_\infty}.$$

Observe that the left and right side of above represent the generating functions for $P_1(n) - 2P_2(n-1)$ and $P_3(3)$, respectively. Hence

$$P_1(n) - 2P_2(n-1) = P_3(n) \quad n > 1.$$

Example 3.2. The following table verifies the case $n = 5$ in Theorem 3.2.

$P_1(5) = 28$	$P_2(4) = 14$	$P_3(5) = 0$
$1_r + 1_r + 1_r + 1_r + 1_r$	$1_r + 1_r + 1_r + 1_r$	
$1_r + 1_r + 1_r + 1_g + 1_g$	$1_r + 1_r + 1_r + 1_g$	
$1_r + 1_r + 1_r + 1_g + 1_g$	$1_r + 1_r + 1_g + 1_g$	
$1_r + 1_r + 1_g + 1_g + 1_g$	$1_r + 1_g + 1_g + 1_g$	
$1_r + 1_g + 1_g + 1_g + 1_g$	$1_g + 1_g + 1_g + 1_g$	
$1_g + 1_g + 1_g + 1_g + 1_g$	$2 + 1_r + 1_r$	
$2 + 1_r + 1_r + 1_r$	$2 + 1_r + 1_g$	
$2 + 1_r + 1_r + 1_g$	$2 + 1_g + 1_g$	
$2 + 1_r + 1_g + 1_g$	$2 + 2$	
$2 + 1_g + 1_g + 1_g$	$3_r + 1_r$	
$2 + 2 + 1_r$	$3_r + 1_g$	
$2 + 2 + 1_g$	$3_g + 1_r$	
$3_r + 1_r + 1_r$	$3_g + 1_g$	
$3_r + 1_r + 1_g$	4	
$3_r + 1_g + 1_g$		
$3_g + 1_r + 1_r$		
$3_g + 1_r + 1_g$		
$3_g + 1_g + 1_g$		
$3_r + 2$		
$3_g + 2$		
$4_r + 1_r$		
$4_r + 1_g$		
$4_g + 1_r$		
$4_g + 1_g$		
$4_w + 1_r$		
$4_w + 1_g$		
5_r		
5_g		

Theorem 3.3. (i) Let $P_1(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm3, \pm5(mod 12)$ with parts congruent to $\pm1, \pm5(mod 12)$ having four colors, and with partition congruent to $\pm3(mod 12)$ having eight colors.

Let $P_2(n)$ denote the number of partitions of n into parts not congruent to $\pm4(mod 12)$, with the parts congruent to $\pm1, \pm5(mod 12)$ having two colors, parts congruent to $\pm2, \pm3(mod 12)$ having four colors and parts congruent to $6(mod 12)$ having eight colors.

Let $P_3(n)$ denote the number of partitions of n into parts congruent to $\pm2, 6(mod 12)$, with parts congruent to $\pm2(mod 12)$ having two colors and parts congruent to $6(mod 12)$ having four colors. Then

$$P_1(n) - 4P_2(n-1) = P_3(n)$$

(ii) Let $P_1(n)$ denote the number of partitions of n into parts not congruent to $\pm 4, 6(mod 12)$, with the parts congruent to $\pm 2(mod 12)$ having eight colors and parts congruent to $\pm 3(mod 12)$ having six colors.

Let $P_2(n)$ denote the number of partitions of n in to parts congruent to $\pm 1, \pm 3, \pm 5(mod 12)$ with five colors. Let $P_3(n)$ denote the number of partitions of n into parts congruent to $\pm 2(mod 12)$ with four colors. Then

$$P_1(n) - P_2(n-1) = P_3(n).$$

(iii) Let $P_1(n)$ denote the number of partitions of n into parts not congruent to $\pm 4, \pm 8, 10(mod 20)$ with parts congruent to $\pm 2, \pm 5, \pm 6(mod 20)$ having four colors. Let $P_2(n)$ denote the number of partitions of n into parts not congruent to $\pm 2, \pm 8(mod 20)$ with parts congruent to $\pm 1, \pm 3, \pm 4, \pm 6, \pm 7, \pm 9(mod 20)$ having 3 colors, and parts congruent to $\pm 5(mod 20)$ having four colors and parts congruent to $10(mod 20)$ having two colors.

Let $P_3(n)$ denote the number of partitions of n into parts congruent to $2(mod 4)$ with two colors. Then

$$P_1(n) - P_2(n-1) = P_3(n).$$

(iv) Let $P_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 3, \pm 5, 7(mod 14)$ with parts $\pm 1, \pm 3, \pm 5(mod 14)$ having three colors and parts congruent to $7(mod 14)$ has 6 colors. Let $P_2(n)$ denote the number of partitions of n into parts not congruent to $\pm 4, \pm 8, \pm 12(mod 28)$ with parts congruent to $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 9, \pm 10, \pm 11, \pm 13(mod 18)$ having two colors, parts congruent to $\pm 7(mod 28)$ having four colors, and parts congruent to $14(mod 28)$ having 6 colors. Let $P_3(n)$ denote the number of partitions of n into partitions congruent to $\pm 2, \pm 6, \pm 10, 14(mod 28)$, with parts congruent to $\pm 2, \pm 6, \pm 10(mod 28)$ having three colors and parts congruent to $14(mod 28)$ having 6 colors. Then

$$P_1(n) - 2P_2(n-1) = P_3(n).$$

Proof. As the pattern of proof of this theorem is identical with our proof of Theorem 3.1 and 3.2. We just give only the references of the required identities.

To prove (i), we employ the identity (2.6).

To prove (ii), we employ the identity (2.7).

To prove (iii), we employ the identity (2.9).

To prove (iv), we employ the identity (2.11).

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