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A SIMPLE PROOF OF TRIPLE PRODUCT IDENTITY OF JACOBI

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Dedicated to Professor G.E. Andrews on his seventieth birthday

Abstract: In this note, we give a simple proof of Jacobi's triple product identity using q-binomial theorem.

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1. Introduction

Jacobi triple product identity states that

$$\sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}} z^n \;=\; (q)_\infty(-zq)_\infty(-1/z)_\infty, \qquad z \neq 0, \quad |q| < 1. \tag{1.1}$$

Andrews [1] gave a proof of (1.1) using two identities of Euler. Combinatorial proofs of Jacobi's triple identity were given by Wright [7], Cheema [2] and Sudler [6]. We can also find a proof of (1.1) in [3]. Hirschhorn [4,5] has proved Jacobi's two-square and four-square theorems using Jacobi's triple product identity. The main purpose of this note is to give a simple proof of (1.1) using only q-binomial theorem:

$$\sum_{n=0}^{\infty} \frac{(a)_n}{q_n} t^n \;=\; \frac{(at)_\infty}{(t)_\infty}, \qquad |t| < 1, \quad |q| < 1. \tag{1.2}$$

Changing a to a/b, t to bt, and letting $b \to 0$ in (1.2), we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{\frac{n(n-1)}{2}}}{(q)_n} t^n \;=\; (at)_\infty, \qquad |q| < 1. \tag{1.3}$$

Putting $a = -1$ in the above identity, we deduce

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q)_n} t^n \;=\; (-t)_\infty, \qquad |q| < 1. \tag{1.4}$$

2. Proof of Jacobi Triple Product Identity

We have

$$\begin{aligned}
\sum_{n=-m}^{\infty} \frac{q^{\frac{n(n+1)}{2}} z^n}{(q^{1+m})_n} &= \sum_{n=0}^{\infty} \frac{q^{\frac{(n-m)(n-m+1)}{2}} z^{n-m}}{(q^{1+m})_{n-m}} \\
&= \frac{q^{\frac{m(m-1)}{2}} z^{-m}}{(q^{1+m})_{-m}} \cdot \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} (zq^{1-m})^n}{(q)_n} \\
&= \frac{q^{\frac{m(m-1)}{2}} z^{-m}}{(q^{1+m})_{-m}} \cdot (-zq^{1-m})_\infty, \quad \text{on using } (1.3), \\
&= \frac{q^{\frac{m(m-1)}{2}} z^{-m}(1+zq^{1-m})(1+zq^{1-m+1})\dots(1+z)(-zq)_\infty}{(q^{1+m})_{-m}} \\
&= q^{\frac{m(m-1)}{2}} (q)_m (-zq)_\infty \left(\frac{1}{z}+q^{1-m}\right)\left(\frac{1}{z}+q^{2-m}\right)\dots\left(\frac{1}{z}+1\right) \\
&= (q)_m (-zq)_\infty (1/z)_m.
\end{aligned}$$

Taking the limit $m \to \infty$, we obtain (1.1).

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