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UNIFIED THEORY OF OBTAINING A NOVEL CLASS OF BILATERAL GENERATING RELATIONS OF CERTAIN SPECIAL FUNCTIONS BY GROUP THEORETIC METHOD

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Abstract : In this paper, we have presented a unified theory of obtaining a novel class of bilateral generating relations involving certain special functions by group theoretic method. In Section 2, the method has been fully discussed and finally we have arrived at a conclusion in connection with the unification of a class of bilateral generating functions involving some special functions which is stated in Theorem 2.1. In Section 3, we have obtained a good number of theorems on bilateral generating functions involving various special functions in course of application of Theorem 2.1 obtained in this investigation.

1. Introduction

Theories in connection with the unification of bilateral or trilateral generating relations for various special functions are of greater importance in the study of special functions. For previous works in this direction, one can see the works [[1]-[7]] and [[8]-[14]] in connection with the unification of bilateral and mixed trilateral generating relations. In this present article, we have discussed a group theoretic method for deriving a unified presentation of a novel class of bilateral generating relations for certain special functions subject to the condition of construction of one parameter continuous transformations group for the special functions under consideration. Furthermore, we would like to mention that a good number of theorems on bilateral generating relations for various special functions can be easily obtained in course of application of our result (Theorem 2.1). In fact the main result of our investigation is given in Section 2(Theorem 2.1).

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2. Group-Theoretic Discussion

We first consider a unilateral generating relation involving a particular special function $p_n^{(\alpha+n)}(x)$ of degree n and of parameter $(\alpha+n)$ as follows:

$$G(x,\ w)=\sum_{n=0}^{\infty} a_n\ p_n^{(\alpha+n)}(x)\ w^n. \tag{2.1}$$

Replacing w by wvz and then multiplying both sides of (2.1) by y^α, we get

$$y^\alpha\ G(x,\ wvz)=\sum_{n=0}^{\infty} a_n\ (p_n^{(\alpha+n)}(x)\ y^\alpha z^n)\ (wv)^n. \tag{2.2}$$

We now suppose that for the above special function $p_n^{(\alpha+n)}(x)$, it is possible to define a linear partial differential operator R, which generates one parameter continuous transformations group as follows:

$$R=A_1(x,y,z)\ \frac{\partial}{\partial x}+A_2(x,y,z)\ \frac{\partial}{\partial y}+A_3(x,y,z)\ \frac{\partial}{\partial z}+A_0(x,y,z) \tag{2.3}$$

such that

$$R(p_n^{(\alpha+n)}(x)\ y^\alpha z^n)=\rho_n\ p_{n+1}^{(\alpha+n)}(x)\ y^{\alpha-1}z^{n+1}, \tag{2.4}$$

where ρ_n is function of $n,\ \alpha$ and independent of x, y and

$$e^{wR}f(x,y,z)=\Omega(x,y,z,w)f\Big(g(x,y,z,w),h(x,y,z,w),k(x,y,z,w)\Big). \tag{2.5}$$

Operating both sides of (2.2) by e^{wR}, we get

$$e^{wR}\Big(y^\alpha\ G(x,\ wvz)\Big)=e^{wR}\Big(\sum_{n=0}^{\infty} a_n\ (p_n^{(\alpha+n)}(x)\ y^\alpha z^n)\ (wv)^n\Big). \tag{2.6}$$

Now the left number of (2.6), with the help of (2.5),becomes

$$\Omega(x,y,z,w)(h(x,y,z,w))^\alpha G\Big(g(x,y,z,w),vwk(x,y,z,w)\Big). \tag{2.7}$$

The right number of (2.6), with the help of(2.4), becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\frac{w^m}{m!}\ \rho_n\ \rho_{n+1}\ \cdots\ \rho_{n+m-1}\ p_{n+m}^{(\alpha+n)}(x)y^{\alpha-m}z^{n+m}(wv)^n.$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}\frac{w^n}{m!}\ \rho_{n-m}\ \rho_{n-m+1}\ \cdots\ \rho_{n-1}\ p_n^{(\alpha+n-m)}(x)\ y^{\alpha-m}z^n v^{n-m}. \tag{2.8}$$

Now equating (2.7) and (2.8) and then putting $y = z = 1$, we get

$$\Omega(x,1,1,w)(h(x,1,1,w))^{\alpha}G(g(x,1,1,w),vwk(x,1,1,w)) = \sum_{n=0}^{\infty} w^n\ \sigma_n(x,\ v),$$

where

$$\sigma_n(x,\ v) = \sum_{m=0}^{n} \frac{a_m}{(n-m)!} \prod_{i=m}^{n-1} \rho_i\ p_n^{(\alpha+m)}(x) v^m.$$

Thus we arrive at the following theorem.

Theorem 2.1. *If*

$$G(x,\ w) = \sum_{n=0}^{\infty} a_n\ p_n^{(\alpha+n)}(x)\ w^n.$$

then

$$\Omega(x,1,1,w)(h(x,1,1,w))^{\alpha}G(g(x,1,1,w),vwk(x,1,1,w)) = \sum_{n=0}^{\infty} w^n\ \sigma_n(x,\ v),$$

where

$$\sigma_n(x,\ v) = \sum_{m=0}^{n} \frac{a_m}{(n-m)!} \prod_{i=m}^{n-1} \rho_i\ p_n^{(\alpha+m)}(x) v^m.$$

3. Applications

We now proceed to give a good number of applications of our result stated in Theorem 2.1.

3.1. **Application-1.** At first we take

$$p_n^{(\alpha+n)}(x) = f_n^{(\beta+n)}(x)\ with\ \alpha = \beta.$$

We now consider the following partial differential operator R :

$$R = xy^{-1}z\ \frac{\partial}{\partial x} - z\ \frac{\partial}{\partial y} - 2z^2y^{-1}\ \frac{\partial}{\partial z} - xy^{-1}z$$

such that

$$R(f_n^{(\beta+n)}(x)\ y^{\beta}z^n) = -(n+1)\ f_{n+1}^{(\beta+n)}(x)\ y^{\beta-1}z^{n+1} \tag{3.1}$$

and

(3.2)

$$e^{wR}f(x,y,z) = exp(-wxy^{-1}z)f\left(x(1+wy^{-1}z), \frac{y}{(1+wy^{-1}z)}, \frac{z}{(1+wy^{-1}z)^2}\right).$$

Comparing (2.4), (2.5) with (3.1), (3.2), we get

$$\rho_n = -(n+1),\ \Omega(x,y,z,w) = exp(-wxy^{-1}z),\ g(x,y,z,w) = x(1+wy^{-1}z),$$

$$h(x,y,z,w) = \frac{y}{(1+wy^{-1}z)},\ k(x,y,z,w) = \frac{z}{(1+wy^{-1}z)^2}.$$

So by the application of our Theorem 2.1, we get the following theorem on bilateral generating relations involving modified Laguerre polynomials.

Theorem 3.1. *If*

$$G(x,\ w) = \sum_{n=0}^{\infty} a_n\ f_n^{(\beta+n)}(x)\ w^n \tag{3.3}$$

then

$$exp(wx)(1-w)^{\beta} G\Big(x(1-w), vw(1-w)^{-2}\Big) = \sum_{n=0}^{\infty} w^n\ \sigma_n(x,\ v), \tag{3.4}$$

where

$$\sigma_n(x,\ v) = \sum_{m=0}^{n} a_m \binom{n}{m} f_n^{(\beta+m)}(x) v^m.$$

3.2. **Application-2.** Now let us take

$$p_n^{(\alpha+n)}(x) = L_{a,\ b,\ (m+n),\ n}(x)\ with\ \alpha = m.$$

Then we consider the following partial differential operator R, where

$$R = bxy^{-1}z\ \frac{\partial}{\partial x} + bz\ \frac{\partial}{\partial y} + 2by^{-1}z^2\ \frac{\partial}{\partial z} - axy^{-1}z$$

such that

$$R(L_{a,\ b,\ (m+n),\ n}(x)\ y^m z^n) = (n+1)\ L_{a,\ b,\ (m+n),\ n+1}(x)\ y^{m-1} z^{n+1} \tag{3.5}$$

and

$$e^{wR} f(x,y,z) = exp\left(\frac{-awxy^{-1}z}{1-bwy^{-1}z}\right) f\Big(x(1-bwy^{-1}z)^{-1}, y(1-bwy^{-1}z)^{-1}, z(1-bwy^{-1}z)^{-2}\Big). \tag{3.6}$$

Comparing (2.4), (2.5) with (3.5), (3.6), we get

$$\rho_n = (n+1),\ \Omega(x,y,z,w) = exp\left(\frac{-awxy^{-1}z}{1-bwy^{-1}z}\right),\ g(x,y,z,w) = x(1-bwy^{-1}z)^{-1},$$

$$h(x,y,z,w) = y(1-bwy^{-1}z)^{-1},\ k(x,y,z,w) = z(1-bwy^{-1}z)^{-2}.$$

So by the application of our Theorem 2.1, we get the following result on bilateral generating relations involving modified Laguerre polynomials.

Theorem 3.2. *If*

$$G(x,\ w) = \sum_{n=0}^{\infty} a_n\ L_{a,\ b,\ (m+n),\ n}(x)\ w^n \tag{3.7}$$

then

$$exp(\frac{-awx}{1-bw})(1-bw)^{-m}G\Big(x(1-bw)^{-1}, vw(1-bw)^{-2}\Big) = \sum_{n=0}^{\infty} w^n\ \sigma_n(x,\ v), \tag{3.8}$$

where

$$\sigma_n(x,\ v) = \sum_{p=0}^{n} a_p\ \binom{n}{p}\ L_{a,\ b,\ (m+p),\ n}(x)v^p.$$

Remark. On specializing the parameters as $a = b = 1$ and $m = 1 + \alpha$ in our Theorem 3.2 we get the following result on Laguerre polynomials.

Theorem 3.3. *If*

$$G(x,\ w) = \sum_{n=0}^{\infty} a_n\ L_n^{(\alpha+n)}(x)\ w^n \tag{3.9}$$

then

$$exp(\frac{-wx}{1-w})(1-w)^{-(1+\alpha)}G\Big(x(1-w)^{-1}, vw(1-w)^{-2}\Big) = \sum_{n=0}^{\infty} w^n\ \sigma_n(x,\ v), \tag{3.10}$$

where

$$\sigma_n(x,\ v) = \sum_{p=0}^{n} a_p\ \binom{n}{p}\ L_n^{(\alpha+p)}(x)v^p,$$

which is found derived in [16].

3.3. **Application-3.** Now we take

$$p_n^{(\alpha+n)}(x) = C_n^{(\lambda+n)}(x)\ with\ \alpha = \lambda.$$

From [15], we see that

$$R = (x^2-1)y^{-1}z\ \frac{\partial}{\partial x} + 2xz\ \frac{\partial}{\partial y} + 3xy^{-1}z^2\ \frac{\partial}{\partial z}$$

such that

$$R(C_n^{(\lambda+n)}(x)\ y^{\lambda}z^n) = (n+1)\ C_{n+1}^{(\lambda+n)}(x)\ y^{\lambda-1}z^{n+1} \tag{3.11}$$

and
(3.12)
$$e^{wR}f(x,y,z) = f\left(\frac{xy-wz}{(wz^2-2wxyz+y^2)^{\frac{1}{2}}}, \frac{y^3}{(wz^2-2wxyz+y^2)}, \frac{y^3z}{(wz^2-2wxyz+y^2)^{\frac{3}{2}}}\right).$$

So by comparing (3.11), (3.12) with (2.4)), (2.5), we get
$$\rho_n = (n+1),\ \Omega(x,y,z,w) = 1,\ g(x,y,z,w) = \frac{xy-wz}{(wz^2-2wxyz+y^2)^{\frac{1}{2}}},$$
$$h(x,y,z,w) = \frac{y^3}{(wz^2-2wxyz+y^2)},\ k(x,y,z,w) = \frac{y^3z}{(wz^2-2wxyz+y^2)^{\frac{3}{2}}}.$$

Then by the application of our Theorem 2.1, we get the following result on bilateral generating relations involving Gegenbauer polynomials.

Theorem 3.4. *If*
$$G(x,\ w) = \sum_{n=0}^{\infty} a_n\ C_n^{(\lambda+n)}(x)\ w^n \tag{3.13}$$

then
$$(w-2wx+1)^{-\lambda}G\left(\frac{x-w}{(w-2wx+1)^{\frac{1}{2}}}, \frac{wv}{(w-2wx+1)^{\frac{3}{2}}}\right) = \sum_{n=0}^{\infty} w^n\ \sigma_n(x,\ v), \tag{3.14}$$

where
$$\sigma_n(x,\ v) = \sum_{m=0}^{n} a_m \binom{n}{m} C_n^{(\lambda+m)}(x)v^m.$$

3.4. **Application-4.** Now we take
$$p_n^{(\alpha+n)}(x) = {}_2F_1(-n,\beta;\alpha+n;x).$$

Then considering the following partial differential operator $R:$
$$R = (1-x)xy^{-1}z\ \frac{\partial}{\partial x} + z\ \frac{\partial}{\partial y} + 2y^{-1}z^2\ \frac{\partial}{\partial z} - \beta xy^{-1}z$$

such that
$$R(\ {}_2F_1(-n,\beta;\alpha+n;x)\ y^\alpha z^n) = (2n+\alpha)\ {}_2F_1(-(n+1),\beta;\alpha+n;x)\ y^{\alpha-1}z^{n+1} \tag{3.15}$$

and
$$e^{wR}f(x,y,z) = (1-wy^{-1}z)^\beta\{1-(1-x)wy^{-1}z\}^{-\beta}$$
$$\times\ f\left(\frac{x}{1-(1-x)wy^{-1}z}, \frac{y}{(1-wy^{-1}z)}, \frac{z}{(1-wy^{-1}z)^2}\right). \tag{3.16}$$

Comparing (3.15), (3.16) with (2.4), (2.5), we get

$$\rho_n = (2n{+}\alpha), \Omega(x,y,z,w) = \frac{(1-wy^{-1}z)^{\beta}}{\{1-(1-x)wy^{-1}z\}^{\beta}}, g(x,y,z,w) = \frac{x}{1-(1-x)wy^{-1}z},$$

$$h(x,y,z,w) = \frac{y}{(1-wy^{-1}z)},\ \ k(x,y,z,w) = \frac{z}{(1-wy^{-1}z)^2}.$$

So by the application of our Theorem 2.1, we at once get the following result on bilateral generating relations involving Hypergeometric polynomials.

Theorem 3.5. *If*

$$G(x,\ w) = \sum_{n=0}^{\infty} a_n\ {}_2F_1(-n,\beta;\alpha+n;x)\ w^n \tag{3.17}$$

then

$$(1-w)^{\beta-\alpha}\{1-(1-x)w\}^{-\beta} G\left(\frac{x}{(1-w(1-x))}, \frac{wv}{(1-w)^2}\right) = \sum_{n=0}^{\infty} w^n\ \sigma_n(x,\ v), \tag{3.18}$$

where

$$\sigma_n(x,\ v) = \sum_{k=0}^{n} a_k \begin{pmatrix} n+\alpha+k-1 \\ 2k+\alpha-1 \end{pmatrix}\ {}_2F_1(-n,\beta;\alpha+k;x)v^k.$$

3.5. **Application-5.** Finally we take

$$p_n^{(\alpha+n)}(x) = Y_n^{\alpha+n}(x;k),$$

k is a non-zero positive integer. Then from [16], we see that

$$R = xy^{-1}z\ \frac{\partial}{\partial x} + z\ \frac{\partial}{\partial y} + (k+1)y^{-1}z^2\ \frac{\partial}{\partial z} + (1-x)y^{-1}z$$

such that

$$R(Y_n^{\alpha+n}(x;k)\ y^{\alpha}z^n) = k(n+1)\ Y_{n+1}^{\alpha+n}(x;k)y^{\alpha-1}z^{n+1} \tag{3.19}$$

and

$$e^{wR}f(x,\ y,\ z) = (1-kwy^{-1}z)^{-\frac{1}{k}}\ exp[\ x\{1-(1-kwy^{-1}z)^{-\frac{1}{k}}\}\ ]$$

$$\times\ f\left(x(1-kwy^{-1}z)^{-\frac{1}{k}}, y(1-kwy^{-1}z)^{-\frac{1}{k}}, z(1-kwy^{-1}z)^{-\frac{k+1}{k}}\right). \tag{3.20}$$

Comparing (3.19), (3.20) with (2.4), (2.5), we get

$$\rho_n = k(n+1),\ \Omega(x,y,z,w) = (1-kwy^{-1}z)^{-\frac{1}{k}}\ exp[x\{1-(1-kwy^{-1}z)^{-\frac{1}{k}}\}],$$

$$g(x,y,z,w) = x(1-kwy^{-1}z)^{-\frac{1}{k}},\ h(x,y,z,w) = y(1-kwy^{-1}z)^{-\frac{1}{k}},$$

$$k(x,y,z,w) = z(1-kwy^{-1}z)^{-\frac{k+1}{k}}.$$

So by the application of our Theorem 2.1, we get the following result on bilateral generating relations involving Konhauser biorthogonal polynomials suggested by the laguerre polynomials.

Theorem 3.6. *If*

(3.21) $$G(x,\ w) = \sum_{n=0}^{\infty} a_n\ Y_n^{\alpha+n}(x;k)\ w^n$$

then

(3.22)

$$(1-kw)^{-\frac{(1+\alpha)}{k}} exp[x\{1-(1-kw)^{-\frac{1}{k}}\}]G(x(1-kw)^{-\frac{1}{k}}, wv(1-kw)^{-\frac{k+1}{k}}) = \sum_{n=0}^{\infty} w^n \sigma_n(x,v),$$

where

$$\sigma_n(x,\ v) = \sum_{m=0}^{n} a_m \begin{pmatrix} n \\ m \end{pmatrix} k^{n-m}\ Y_n^{\alpha+m}(x;k)v^m,$$

which is found derived in [16].

Remark. On putting $k=1$ in our Theorem 3.6 we get the Theorem 3.3.

4. Conclusion

From the above discussion, it is clear that one may apply Theorem 2.1 in the case of other polynomials and functions existing in the field of special functions subject to the condition of construction of one parameter continuous transformations group for the said special function(s). Furthermore, the importance of the above theorems (3.1-3.6) lies in the fact that whenever one knows a unilateral generating function of the form (3.3,3.7 etc.) then the corresponding bilateral generating function can at once be written down from (3.4,3.8 etc.). So one can get a large number of bilateral generating functions by attributing different suitable values to a_n in (3.3,3.7etc.).

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