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ON AN EXAMPLE RELATED TO A CONJECTURE OF RAJENDRA BHATIA AND PETER ŠEMRL

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Dedicated to Professor G.E. Andrews on his seventieth birthday

Abstract: Suppose A and B are two linear operators on C^n with a non-Euclidean norm. In a paper [1] on orthogonality of matrices Bhatia and Šemrl conjectures that $\|A\| \leq \|A + \lambda B\|$ for all $\lambda \in C$ iff there exists a unit vector $\tilde{z} \in C^n$ such that $\|A\tilde{z}\| = \|A\|$ and $\|A\tilde{z}\| \leq \|(A + \lambda B)\tilde{z}\|$ for all $\lambda \in C$. The conjecture was negated by Li [2]. We here give an easy example to negate a slightly modified form of the the conjecture $\|A\| < \|A+\lambda B\|$ for all non-zero scalar $\lambda \in C$ iff there exists a unit vector $\tilde{z} \in C^n$ such that $\|A\tilde{z}\| = \|A\|$ and $\|A\tilde{z}\| < \|(A + \lambda B)\tilde{z}\|$ for all non-zero scalar $\lambda \in C$.

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1. Introduction

Clearly one part of the modified form of the conjecture is always true i.e., if there exists a unit vector $\tilde{z} \in C^n$ such that $\|A\tilde{z}\| = \|A\|$ and $\|A\tilde{z}\| < \|(A + \lambda B)\tilde{z}\|$for all non-zero scalar $\lambda \in C$ then $\|A\| < \|A + \lambda B\|$ for all non-zero scalar $\lambda \in C$. In fact this part is true for the original conjecture. We give an example to show that the other part is not always true.

Consider C^n with the norm $\|.\|_\infty$ defined as $\|\tilde{z}\|_\infty = \max\{|z_1|, |z_2| \dots |z_n|\}$, where

$$\tilde{z} = \begin{pmatrix} z_1 \\ z_2 \\ . \\ z_n \end{pmatrix} \in C^n$$

Let us consider the linear operator

$$A = \begin{pmatrix} 1 & 1 & . & . & . & 1 & 1 \\ 1 & 1 & . & . & . & 1 & 0 \\ 1 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 1 & 1 & 0 & . & . & 0 & 0 \\ n^2 & 0 & . & . & . & 0 & 0 \end{pmatrix}$$

i.e. $A : C^n \to C^n$ is given by

$$A\tilde{z} = (\overbrace{z_1 + z_2 + \ldots + z_n}, \overbrace{z_1 + z_2 + \ldots + z_{n-1}}, \ldots, \overbrace{z_1 + z_2}, \overbrace{n^2 z_1})$$

Then for all $\lambda \in C$, we have

$$(A+\lambda I)\tilde{z} = (\overbrace{z_1(1+\lambda) + z_2 + .. + z_n}, \overbrace{z_1 + z_2(1+\lambda) + .. + z_{n-1}}, .., \overbrace{z_1 + z_2 + \lambda z_{n-1}}, \overbrace{n^2 z_1 + \lambda z_n})$$

and so

$$\|A+\lambda I\| = \sup_{\|\tilde{z}\|_\infty = 1} \|(A+\lambda I)\tilde{z}\| = \sup_{\|\tilde{z}\|_\infty = 1} \max\{|z_1(1+\lambda)+z_2+\ldots+z_n|,$$

$$|z_1 + z_2(1+\lambda) + \ldots + z_{n-1}|, \ldots |z_1 + z_2 + \lambda z_{n-1}|, |n^2 z_1 + \lambda z_n|\} = n^2 + |\lambda|,$$

which is attained at $\tilde{z} = (e^{i\phi}, z_2, z_3, \ldots, z_{n-1}, e^{-i\theta})$, where $\lambda = |\lambda| e^{i\theta}$ and $|z_2| \leq 1,\ |z_3| \leq 1, \ldots |z_{n-1}| \leq 1$. Thus $\|A\| < \|A + \lambda I\|$ for all $0 \neq \lambda \in C$.

We now show that there exists no such vector $\tilde{z} \in C^n$ such that $\|A\tilde{z}\|_\infty = \|A\|$ and $\|A\tilde{z}\|_\infty < \|(A + \lambda I)\tilde{z}\|_\infty$ for all $0 \neq \lambda \in C$. Clearly $\|A\|$ is attained at those unit vectors $\tilde{z}$ for which $\tilde{z} = (e^{i\phi}, z_2, \ldots z_n)$ where $|z_2| \leq 1,\ |z_3| \leq 1, \ldots |z_n| \leq 1$. If possible suppose there exists $\tilde{z} = (e^{i\phi},\ z_2, \ldots z_{n-1}, |z_n| e^{i\theta})$ where $|z_2| \leq 1,\ |z_3| \leq 1, \ldots |z_n| \leq 1$ such that $\|A\tilde{z}\|_\infty < \|(A + \lambda I)\tilde{z}\|_\infty$ for all $0 \neq \lambda \in C$.

Choose $\lambda_0 = e^{-i(\theta - \phi + \pi)}$. Then $\|(A + \lambda_0 I)\tilde{z}\|_\infty = n^2 - |z_n| \leq \|A\tilde{z}\|_\infty = n^2$ which is a contradiction.

This example negates the above mentioned conjecture which states that

Suppose A and B are two linear operators on C^n with a non-Euclidean norm. Then $\|A\| < \|A + \lambda I\|$ for all non-zero $\lambda \in C$ iff there exists a unit vector $\tilde{z} \in C^n$ such that $\|A\tilde{z}\|_\infty = \|A\|$ and $\|A\tilde{z}\|_\infty < \|(A + \lambda I)\tilde{z}\|_\infty$ for all non-zero $\lambda \in C$.

References

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