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# N-THETA FUNCTION IDENTITIES

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*Dedicated to Professor G. E. Andrews on his seventieth birthday*

**Abstract:** Ramanujan develops, in Chapter 16 of his second notebook, the theory of theta-function and recorded several identities without proofs. All these have been proved by Adiga, Berndt, Bhargava and Watson. In this paper, we establish several results of N-theta function which are analogous to the results in the Entries in Chapter 16 of Ramanujan's second notebook.

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## 1. Introduction

Ramanujan develops, in Chapter 16 of his second notebook [3], the theory of theta-function and his theta-function is defined by

$$f(a,b) \;=\; \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab|<1.$$

Following Ramanujan, we define

$$\varphi(q) := f(q,q) \;=\; \sum_{n=-\infty}^{\infty} q^{n^2} \tag{1.1}$$

$$\psi(q) := f(q,q^3) \;=\; \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \tag{1.2}$$

$$f(-q) := f(-q,-q^2) \;=\; \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}. \tag{1.3}$$

Following Ramanujan, we define a new N- Theta function by

$$f_N(a,b) \;=\; \sum_{k=-\infty}^{\infty} a^{\frac{k^N(k^N+1)}{2}} b^{\frac{k^N(k^N-1)}{2}}, \;\; |ab|<1. \tag{1.4}$$

If we set $a = qe^{2iz}$, $b = qe^{-2iz}$ and $q = e^{\pi i\tau}$, where $z$ is complex and $Im(\tau) > 0$ in (1.4), then we deduce that

$$\vartheta_{N,3}(z,\tau) = \sum_{k=-\infty}^{\infty} q^{k^{2N}} e^{2ik^N z}, \tag{1.5}$$

where $\vartheta_{N,3}(z,\tau)$ is similar to one of the classical theta-functions in its standard notation [4]. Now we obtain special cases of $f_N(a,b)$ which are similar to the special cases (1.1)-(1.3) of Ramanujan's theta-function $f(a,b)$.

$$\varphi_N(q) := f_N(q,q) = \sum_{k=-\infty}^{\infty} q^{k^{2N}} \tag{1.6}$$

$$\psi_N(q) = \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+\frac{1}{2})^{2N}} \tag{1.7}$$

and

$$f_N(-q) := f_N(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k^N(3k^N-1)}{2}}. \tag{1.8}$$

**Remark 1.1** Putting $N = 1$ in (1.4), then we obtain 1-Theta function which is same as Ramanujan's $f(a,b)$.

## 2. Some Results Similar to Ramanujan's Theta-functions

In this section, we establish several results which are similar to the results in the Entries in Chapter 16 of his second notebook [3].

**Theorem 2.1.** We have

$$f_N(a,b) = f_N(b,a) \quad \text{if } N \text{ is odd} \tag{2.1}$$

**Proof.** Replacing $k$ by $-k$ in (1.4), we obtained the required result.

**Theorem 2.2.** We have

$$\varphi_N(q) + \varphi_N(-q) = 2\varphi_N(q^{4^N}). \tag{2.2}$$

**Proof.** Using (1.5), we find that

$$\begin{aligned}\varphi_N(q)+\varphi_N(-q) &= \sum_{k=-\infty}^{\infty} q^{k^{2N}} + \sum_{k=-\infty}^{\infty} (-1)^k q^{k^{2N}} \\ &= \sum_{k=-\infty}^{\infty} q^{k^{2N}}(1+(-1)^k) \\ &= 2\sum_{k=-\infty}^{\infty} q^{(2k)^{2N}} \\ &= 2\varphi_N(q^{4^N}).\end{aligned}$$

**Corollary 2.1.** We have

$$\varphi(q)+\varphi(-q) \;=\; 2\varphi(q^4). \tag{2.3}$$

**Proof.** Putting $N=1$ in (2.2), we obtain (2.3)

**Remark 2.1.** The identity (2.3) is same as the identity in Entry 25(i) of Chapter 16 of Ramanujan's second notebook [3].

**Theorem 2.3.** We have

$$\varphi_N(q)-\varphi_N(-q) \;=\; 4\psi_N(q^{2^{2N+1}}). \tag{2.4}$$

**Proof.** Using (1.5), we find that

$$\begin{aligned}\varphi_N(q)-\varphi_N(-q) &= \sum_{k=-\infty}^{\infty} q^{k^{2N}} - \sum_{k=-\infty}^{\infty} (-1)^k q^{k^{2N}} \\ &= \sum_{k=-\infty}^{\infty} q^{k^{2N}}(1-(-1)^k) \\ &= 2\sum_{k=-\infty}^{\infty} q^{(2k+1)^{2N}} \\ &= 4\psi_N(q^{2^{2N+1}}).\end{aligned}$$

**Corollary 2.2.** We have

$$\varphi(q)-\varphi(-q) \;=\; 4q\psi(q^8). \tag{2.5}$$

**Proof.** Putting $N = 1$ in (2.6), we obtain the require result.

**Remark 2.2.** The identity (2.5) is same as the identity in Entry 25(ii) of Chapter 16 of Ramanujan's second notebook [3].

**Theorem 2.4.** We have

$$\varphi_N^2(q) + \varphi_N^2(-q) = 2\varphi_N^2(q^{4^N}) + 8\psi_N^2(q^{2^{2N+1}}). \tag{2.6}$$

**Proof.** Using (1.5), we find that

$$\begin{aligned}\varphi_N^2(q) + \varphi_N^2(-q) &= \sum_{m,k=-\infty}^{\infty} q^{m^{2N}+k^{2N}}(1 + (-1)^{m+k}) \\ &= \sum_{m,k \text{ both even}} q^{m^{2N}+k^{2N}}(1 + (-1)^{m+k}) \\ &+ \sum_{m,k \text{ both odd}} q^{m^{2N}+k^{2N}}(1 + (-1)^{m+k}) \\ &= 2\varphi_N^2(q^{4^N}) + 8\psi_N^2(q^{2^{2N+1}}).\end{aligned}$$

**Corollary 2.3.** We have

$$\varphi^2(q) + \varphi^2(-q) \;=\; 2\varphi^2(q^2). \tag{2.7}$$

**Proof.** Putting $N = 1$ in (2.6), we obtain (2.7).

**Remark 2.3.** The identity (2.9) is same as the identity in Entry 25(vi) of Chapter 16 of Ramanujan's second notebook [3].

**Theorem 2.5.** We have

$$\varphi_N^2(q) - \varphi_N^2(-q) \;=\; 8\varphi_N(q^{4^N})\psi_N(q^{2^{2N+1}}). \tag{2.8}$$

Proof of (2.8) is similar to the proof of (2.6). So we omit the details.

**Corollary 2.4.** We have

$$\varphi^2(q) - \varphi^2(-q) \;=\; 8\varphi(q^4)\psi(q^8). \tag{2.9}$$

**Proof.** Putting $N = 1$ in (2.8), we obtain (2.9).

**Remark 2.4.** The identity (2.9) is same as the identity in Entry 25(v) of Chapter 16 of Ramanujan's second notebook [3].

**Theorem 2.6.** We have

$$f_N(a,b) + f_N(-a,-b) \;=\; 2f_N\left(a^{\frac{2^{2N}+2^N}{2}}b^{\frac{2^{2N}-2^N}{2}}, a^{\frac{2^{2N}-2^N}{2}}b^{\frac{2^{2N}+2^N}{2}}\right). \qquad (2.10)$$

**Proof.** Using (1.4), we deduce that

$$\begin{aligned}
f_N(a,b) + f_N(-a,-b) &= \sum_{k=-\infty}^{\infty} a^{\frac{k^N(k^N+1)}{2}} b^{\frac{k^N(k^N-1)}{2}} (1+(-1)^k) \\
&= 2\sum_{k=-\infty}^{\infty} a^{\frac{(2k)^N((2k)^N+1)}{2}} b^{\frac{(2k)^N((2k)^N-1)}{2}} \\
&= 2\sum_{k=-\infty}^{\infty} ((ab)^{2^{2N}})^{\frac{k^{2N}-k^N}{2}} (a^{-1}b)^{\frac{(2^{2N}-2^N)k^N}{2}} (a^{2^{2N}})^{k^N} \\
&= 2\sum_{k=-\infty}^{\infty} ((ab)^{4^k})^{\frac{k^{2N}-k^N}{2}} (a^{\frac{2^{2N}+2^N}{2}} b^{\frac{2^{2N}-2^N}{2}})^{k^N} \\
&= 2f_N\left(a^{\frac{2^{2N}+2^N}{2}}b^{\frac{2^{2N}-2^N}{2}}, a^{\frac{2^{2N}-2^N}{2}}b^{\frac{2^{2N}+2^N}{2}}\right).
\end{aligned}$$

**Corollary 2.5.** We have

$$f(a,b) + f(-a,-b) \;=\; 2f(a^3b, ab^3). \qquad (2.11)$$

**Proof.** Putting $N = 1$ in (2.10), we obtain (2.11).

**Remark 2.5.** The identity (2.11) is same as the identity in Entry 30(ii) of Chapter 16 of Ramanujan's second notebook [3].

## 3. Some Results Similar to Classical Theta-functions

The theta-function were first systematically studied by Jacobi, who obtained their properties by purely algebraical methods. The four types of theta functions are as follows:

$$\vartheta_1(z,q) \;=\; 2\sum_{k=0}^{\infty}(-1)^k q^{(k+\frac{1}{2})^2}\sin(2k+1)z, \qquad (3.1)$$

$$\vartheta_2(z,q) \;=\; 2\sum_{k=0}^{\infty} q^{(k+\frac{1}{2})^2}\cos(2k+1)z, \qquad (3.2)$$

$$\vartheta_3(z,q) \;=\; 2\sum_{k=-\infty}^{\infty} q^{k^2}\cos 2kz \qquad (3.3)$$

and

$$\vartheta_4(z,q) \;=\; \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} \cos 2kz. \tag{3.4}$$

In this section, we introduce generalized classical theta-functions as follows:

$$\vartheta_{N,1}(z,q) \;=\; \sum_{k=0}^{\infty} q^{k^{2N}} \sin 2k^N z, \tag{3.5}$$

$$\vartheta_{N,2}(z,q) \;=\; \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}(k+\frac{1}{2})^{2N}} e^{2i(2k+1)^N z}, \tag{3.6}$$

$$\vartheta'_{N,2}(z,q) \;=\; \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+\frac{1}{2})^{2N}} \cos 2(2k+1)^N z, \tag{3.7}$$

$$\vartheta_{N,3}(z,q) \;=\; \sum_{k=-\infty}^{\infty} q^{k^{2N}} e^{2ik^N z}, \tag{3.8}$$

$$\vartheta'_{N,3}(z,q) \;=\; \sum_{k=0}^{\infty} q^{k^{2N}} \cos 2k^N z, \tag{3.9}$$

and

$$\vartheta_{N,4}(z,q) \;=\; \sum_{k=-\infty}^{\infty} (-1)^k q^{k^{2N}} e^{2ik^N z}. \tag{3.10}$$

**Theorem 3.1.** We have

$$\vartheta_{N,3}(z,q) + \vartheta_{N,4}(z,q) \;=\; 2\vartheta_{N,3}(2^N z, q^{2^{2N}}). \tag{3.11}$$

**Proof.** Using (3.8) and (3.10), we find that

$$\begin{aligned}\vartheta_{N,3}(z,q) + \vartheta_{N,4}(z,q) &= \sum_{k=-\infty}^{\infty} q^{k^{2N}} e^{2ik^N z}(1+(-1)^k) \\ &= 2\sum_{k=-\infty}^{\infty} (q^{2^{2N}})^{k^{2N}} e^{2ik^N(2^N z)} \\ &= 2\vartheta_{N,3}(2^N z, q^{2^{2N}}).\end{aligned}$$

**Corollary 3.1.** We have

$$\vartheta_{N,3}(0,q) + \vartheta_{N,4}(0,q) \;=\; 2\vartheta_{N,3}(0, q^{2^{2N}}). \tag{3.12}$$

**Proof.** Putting $z = 0$ in (3.11), we obtain (3.12).

**Remark 3.1.** The identity (3.12) is same as (2.2).

**Theorem 3.2.** We have

$$\vartheta_{N,3}(z,q) - \vartheta_{N,4}(z,q) \;=\; \begin{cases} 4\vartheta_{N,2}(z,q^{2^{2N+1}}) & \text{if } N \text{ is even} \\ 4\vartheta'_{N,2}(z,q^{2^{2N+1}}) & \text{if } N \text{ is odd.} \end{cases} \tag{3.13}$$

**Proof.** Using (3.8) and (3.10), we find that

$$\begin{aligned}
\vartheta_{N,3}(z,q) - \vartheta_{N,4}(z,q) &= \sum_{k=-\infty}^{\infty} q^{k^{2N}} e^{2ik^N z}(1-(-1)^k) \\
&= 2\sum_{k=-\infty}^{\infty} (q^{2^{2N+1}})^{\frac{1}{2}(k+\frac{1}{2})^{2N}} e^{2i(2k+1)^N z} \\
&= 2\sum_{k=0}^{\infty} (q^{2^{2N+1}})^{\frac{1}{2}(k+\frac{1}{2})^{2N}} e^{2i(2k+1)^N z} \\
&+ 2\sum_{k=0}^{\infty} (q^{2^{2N+1}})^{\frac{1}{2}(k+\frac{1}{2})^{2N}} e^{2i(-1)^N(2k+1)^N z} \\
&= 2\sum_{k=0}^{\infty} (q^{2^{2N+1}})^{\frac{1}{2}(k+\frac{1}{2})^{2N}} \left[e^{2i(2k+1)^N z} + e^{2i(-1)^N(2k+1)^N z}\right] \\
&= \begin{cases} 4\sum_{k=0}^{\infty} (q^{2^{2N+1}})^{\frac{1}{2}(k+\frac{1}{2})^{2N}} e^{2i(2k+1)^N z} & \text{if } N \text{ is even} \\ 4\sum_{k=0}^{\infty} (q^{2^{2N+1}})^{\frac{1}{2}(k+\frac{1}{2})^{2N}} \cos 2(2k+1)^N z & \text{if } N \text{ is odd} \end{cases} \\
&= \begin{cases} 4\vartheta_{N,2}(z,q^{2^{2N+1}}) & \text{if } N \text{ is even} \\ 4\vartheta'_{N,2}(z,q^{2^{2N+1}}) & \text{if } N \text{ is odd.} \end{cases}
\end{aligned}$$

**Corollary 3.2.** We have

$$\vartheta_{N,3}(0,q) - \vartheta_{N,4}(0,q) \;=\; 4\vartheta_{N,2}(0,q^{2^{2N+1}}). \tag{3.14}$$

**Proof.** Putting $z = 0$ in (3.13), we obtain

**Remark 3.2.** The identity (3.14) is same as (2.4).

**Theorem 3.3.** We have

$$\vartheta_{N,3}(z,q) + \vartheta_{N,3}(-z,q) \;=\; 4\vartheta'_{N,3}(z,q) - 2. \tag{3.15}$$

**Proof.** Using (3.8), we find that

$$\begin{aligned}\vartheta_{N,3}(z,q)+\vartheta_{N,3}(-z,q) &= \sum_{k=-\infty}^{\infty} q^{k^{2N}}\left(e^{2ik^N z}+e^{-2ik^N z}\right)\\ &= 2\sum_{k=-\infty}^{\infty} q^{k^{2N}} cos 2k^N z\\ &= 2\left(1+2\sum_{k=1}^{\infty} q^{k^{2N}} cos 2k^N z\right)\\ &= 4\vartheta'_{N,3}(z,q)-2.\end{aligned}$$

**Theorem 3.4.** We have

$$\vartheta_{N,3}(z,q)-\vartheta_{N,3}(-z,q) \;=\; 4i\vartheta_{2N,1}(z,q). \tag{3.16}$$

Proof of (3.16) is samiliar to the proof of (3.15). So we omit the details.

**Theorem 3.5.** We have

$$\vartheta_{N,3}(z,q) \;=\; \vartheta_{N,3}(2^N z, q^{2^{2N}})+\vartheta_{N,2}(z,q^{2^{2N+1}}), \tag{3.17}$$

$$\vartheta_{N,4}(z,q) \;=\; \vartheta_{N,3}(2^N z, q^{2^{2N}})-\vartheta_{N,2}(z,q^{2^{2N+1}}), \tag{3.18}$$

$$\vartheta^2_{N,3}(2^N z, q^{2^{2N}})-\vartheta^2_{N,2}(z,q^{2^{2N+1}}) \;=\; \vartheta_{N,3}(z,q)\vartheta_{N,4}(z,q), \tag{3.19}$$

and

$$\vartheta^2_{N,3}(z,q)-\vartheta^2_{N,4}(z,q)=\begin{cases}4\vartheta_{N,3}(2^N z, q^{2^{2N}})\vartheta_{N,2}(z,q^{2^{2N+1}}) & \text{if } N \text{ is even}\\[2ex] 4\vartheta_{N,3}(2^N z, q^{2^{2N}})\vartheta'_{N,2}(z,q^{2^{2N+1}}) & \text{if } N \text{ is odd.}\end{cases} \tag{3.20}$$

Proofs of the identities (3.17)-(3.20) follows very easily from the definitions (3.8) and (3.10). So we omit the details.

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