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UNEVEN FROBENIUS REPRESENTATIONS OF PARTITIONS USING 3 AND 4 COLORS

Louis W. Kolitsch*

Department of Mathematics and Statistics
University of Tennessee at Martin, Martin, TN 38237, U.S.A.
E-mail: lkolitsc@utm.edu

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Dedicated to George E. Andrews on the occasion of his 70th birthday

Abstract: In this paper we will look at some Frobenius representations where the top and bottom rows are not the same length. These uneven Frobenius representations are associated with the generating functions for 3- and 4-color generalized Frobenius partitions. Surprisingly, we get representations for some easily described partitions.

Keywords and Phrases: 3- and 4-colors generalized Frobenius partitions, uneven Frobenius representations, generating functions
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1. Introduction

In 1985 Andrews introduced generalized Frobenius partitions [1]. The generating function for generalized Frobenius partitions with k colors is given by the coefficient of z^0 in $(z;q)_\infty^k(z^{-1}q;q)_\infty^k$ and a typical partition has the form $\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_r \\ b_1 & b_2 & b_3 & \cdots & b_r \end{pmatrix}$ where the number n represented by this symbol is given by $n = r + \sum_{i=1}^{r}(a_i + b_i)$ and the entries in each row are distinct and come from k copies of the nonnegative integers distinguished by k colors. The exponent on z keeps track of the difference in the number of entries in the top and bottom rows of the Frobenuis symbol.

In this paper we will look at representations of the form $\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s & \end{pmatrix}$ where the number of entries in the top row exceeds the number of entries in the bottom row; that is, $r > s$. The number n represented by this symbol will be given by $n = s + \sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i$. Specifically, we will consider the case where $r = s + 2$ and the entries in each row are distinct and come from 3 copies of the

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nonnegative integers distinguished by 3 colors and the case where $r = s + 1$ and the entries in each row are distinct and come from 4 copies of the nonnegative integers distinguished by 4 colors. The first case will be called Type I partitions and the second case will be called Type II partitions. The results we will prove about these two types of partitions are given in the following theorem.

2. Main Results

Theorem 2.1.

(i) The number of Type I partitions of n is three times the number of partitions of n into parts where the parts not divisible by three can appear in four colors and the parts divisible by three can appear in only one color.

(ii) The number of Type II partitions of n is four times the number of partitions of n into parts where the odd parts can appear in seven colors and the even parts can appear in only one color.

We will prove this theorem analytically by rewriting the generating function for each type of uneven Frobenius partitions in the appropriate form. We will start with the Type I partitions. The generating function for the Type I partitions is given by the coefficient of z^2 in

$$(z;q)_\infty^3 (z^{-1}q;q)_\infty^3. \tag{2.1}$$

Using Jacobi's Triple Product Identity this generating function can be written as the coefficient of z^2 in

$$\frac{1}{(q;q)_\infty^3}\Big(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m^2-m}{2}}\Big)^3. \tag{2.2}$$

The coefficient of z^2 is given by

$$\frac{1}{(q;q)_\infty^3}\sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty} q^{\frac{1}{2}(a^2+b^2+(2-a-b)^2)-1} \tag{2.3}$$

which can be rewritten as

$$\frac{1}{(q;q)_\infty^3}\sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty} q^{a^2+b^2+ab+a+b} \tag{2.4}$$

after we shift the indices for a and b by one unit. This generating function can be rewritten as

$$\frac{1}{q^{\frac{1}{3}}(q;q)_\infty^3}\sum_{a=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} q^{(a+\frac{1}{3})^2+(b+\frac{1}{3})^2+(a+\frac{1}{3})(b+\frac{1}{3})}. \tag{2.5}$$

By a result of Borwein *et al.* [2] this is equal to

$$\frac{3(q^3;q^3)_\infty^3}{(q;q)_\infty^4} \tag{2.6}$$

which is the desired generating function.

The generating function for the Type II partitions is given by the coefficient of z in

$$(z;q)_\infty^4(z^{-1}q;q)_\infty^4. \tag{2.7}$$

Using Jacobi's Triple Product Identity this generating function can be written as the coefficient of z in

$$\frac{1}{(q;q)_\infty^4}\Big(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m^2-m}{2}}\Big)^4. \tag{2.8}$$

The coefficient of z is given by

$$\frac{1}{(q;q)_\infty^4}\sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty}\sum_{c=-\infty}^{\infty} q^{\frac{1}{2}(a^2+b^2+c^2+(1-a-b-c)^2-1)} \tag{2.9}$$

which can be rewritten as

$$\frac{1}{(q;q)_\infty^4}\sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty}\sum_{c=-\infty}^{\infty} q^{a^2+b^2+c^2-a-b-c+ab+ac+bc}. \tag{2.10}$$

By an identity in [3] this generating function is equal to

$$\frac{1}{(q;q)_\infty^4}\Big(\frac{1}{2}\sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty}\sum_{c=-\infty}^{\infty} q^{(\frac{a^2+a}{2}+\frac{b^2+b}{2}+\frac{c^2+c}{2})}\Big) = \frac{1}{2(q;q)_\infty^4}\Big(\sum_{m=-\infty}^{\infty} q^{\frac{m^2+m}{2}}\Big)^3. \tag{2.11}$$

Applying Jacobi's Triple Product Identity this becomes

$$\frac{1}{2(q;q)_\infty^4}((-1;q)_\infty(-q;q)_\infty(q;q)_\infty)^3 = \frac{4(q^2;q^2)_\infty^6}{(q;q)_\infty^7} \tag{2.12}$$

which is the desired generating function.

Having verified the results analytically, it would be nice if we could verify the results combinatorially. Though I do not have a combinatorial proof of the complete theorem, I can explain the presence of the factor of 3 for the Type I partitions and the presence of the factor of 4 for the Type II partitions. Starting with a partition of n of either type you can cyclically permute the colors on the parts to get another partition of n. Because of the particular unevenness of the rows in the Frobenius representations, the order of a Type I partition under a cyclic permutation of the colors will be 3 and the order of a Type II partition will be 4. Hence the Type I partitions can be grouped in subsets of size 3 and the Type II partitions can be grouped in subsets of size 4.

I have checked the data for $k < 9$ colors and no similar results appear to exist. In conclusion I leave two challenges for the reader: (1) Find similar results for other values of k colors; and (2) Find a combinatorial proof of the theorem in this paper.

References

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