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ON EULER TYPE INTEGRALS

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Dedicated to Professor G.E. Andrews on his seventieth birthday

Abstract: In this paper, we establish a theorem connecting Euler type single and double integrals. We derive a number of new results as application of the theorem.

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1. Introduction

We recall the Euler integral which defines the beta function

$$B(\alpha,\beta) \;=\; \int_0^1 u^{\alpha-1}(1-v)^{\beta-1}du \;=\; \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \;\; \mathrm{Re}(\alpha),\; \mathrm{Re}(\beta)>0 \qquad (1.1)$$

and a connection between single and double Eulerian integral

$$\int_0^1\int_0^1 f(xy)(1-x)^{\alpha-1}y^{\alpha}(1-y)^{\beta-1}\,dxdy \;=\; B(\alpha,\beta)\int_0^1 f(t)(1-t)^{\alpha+\beta-1}dt \quad (1.2)$$

which is a special case of the result [5, p.379,4.2.4(1)] for $c=0$.

Recently Ismail and Pitman [3] obtained explicit evaluations of some integrals of Euler's type

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}f(u)du$$

for some particular functions f, specially in the symmetric case $\alpha=\beta$. Khan *et al.* in [4] extended further these results to generalize the evaluations of certain Euler type integrals.

Motivated and inspired by the result (1.2), the work of Ismail and Pitman [3] and Khan, Agarwal, Pathan and Mohammad [4], in this paper, we obtain a theorem on Euler type integrals and apply it to obtain the explicit evaluations of certain Eulerian integrals.

Just to give an idea of connection between single and double Eulerian integrals, we consider a double Eulerian integral

$$I = \int_0^1 \int_0^1 \Big[\Big(\frac{1-x}{1-xy}\Big)y\Big]^\alpha \Big(\frac{1-y}{1-xy}\Big)^\beta \frac{(1-xy)}{(1-x)(1-y)} \times \Big(1 - t\Big[\Big(\frac{1-x}{1-xy}\Big)y\Big]^m \Big(\frac{1-y}{1-xy}\Big)^M\Big)^{-1} dxdy \tag{1.3}$$

which on using the expression

$$(1-X)^{-1} = \sum_{n=0}^{\infty} X^n$$

and a result [1, p.445]

$$\int_0^1 \int_0^1 \Big[\Big(\frac{1-x}{1-xy}\Big)y\Big]^\alpha \Big(\frac{1-y}{1-xy}\Big)^\beta \Big(\frac{1-xy}{(1-x)(1-y)}\Big) dxdy = B(\alpha,\beta), \quad \text{Re}(\alpha),\ \text{Re}(\beta) > 0 \tag{1.4}$$

becomes

$$I = B(\alpha,\beta) \sum_{n=0}^{\infty} \frac{t^n\,(\alpha)_{nm}\,(\beta)_{nM}}{(\alpha+\beta)_{nm+nM}}. \tag{1.5}$$

On comparison, we can find that this result is equivalent to a recent result [4, p.2000(4.9)] in the form of Eulerian integral of single variable

$$I = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(1-tu^m(1-u)^M)^{-1} du \tag{1.6}$$

Further, taking m and M, positive integers, the above integrals (1.3) or (1.6) can be evaluated in terms of Gaussian hypergeometric function.

In the following section, we will see how the above results can be extended to more generalized forms of double and single Eulerian integrals

2. Theorem on Eulerian Integrals

Consider a three-variable generating function $F(X,Y,t)$ which possesses a formal (not necessarily convergent for $T \neq 0$) power series expansion in T such that

$$F(X,Y,T) = \sum_{n=0}^{\infty} C_n \Phi_n(X,Y) T^n \tag{2.1}$$

where the set $\{\Phi_n(X,Y)\}_{n=0}^{\infty}$ is independent of T and the coefficient set $\{c_n\}_{n=0}^{\infty}$ may contain the parameters of the set $\{\Phi_n(X,Y)\}_{n=0}^{\infty}$ but is independent of X and T.

Theorem 2.1. Let the generating function $F(X,Y,t)$ defined by eq. (2.1) be such that $F(X,Y,t(1-x)^{\lambda}y^{\lambda}(1-y)^{\mu})$ remains uniformly convergent for $x,y\in(0,1)$, $\mu,\lambda\geq 0$ and $\mu+\lambda>0$. Then

$$\int_0^1\int_0^1 f(xy)(1-x)^{\alpha-1}y^{\alpha}(1-y)^{\beta-1}F\Big(X,Y,T(1-x)^{\lambda}y^{\lambda}(1-y)^{\mu}\Big)\,dxdy$$
$$= \sum_{n=0}^{\infty}B(\alpha+n\lambda,\beta+n\mu)c_n\Phi_n(X,Y)\int_0^1 f(t)(1-t)^{\alpha+\beta+n\lambda+n\mu-1}\,dt \qquad (2.2)$$

Proof. Applying the definition of $F(X,Y,t)$ given in eq. (2.1) in the L.H.S. of (2.2), changing the order of integrations and summation, we get

$$\sum_{n=0}^{\infty}c_n\Phi_n(X,Y)T^n\int_0^1\int_0^1 f(xy)(1-x)^{\alpha+n\lambda-1}y^{\alpha+n\lambda}(1-y)^{\beta+n\mu-1}dxdy \qquad (2.3)$$

which on using (1.2) yields the R.H.S. of (2.2).

Corollary 2.1. With definition (2.1), we have

$$\int_0^1\int_0^1(1-x)^{\alpha-1}y^{\alpha}(1-y)^{\beta-1}F\Big(X,Y,T(1-x)^{\lambda}y^{\lambda}(1-y)^{\mu}\Big)\,dxdy$$
$$= \sum_{n=0}^{\infty}c_n\Phi_n(X,Y)T^nB(\alpha+n\lambda,\beta+n\mu) \qquad (2.4)$$

Proof. Taking $f(t)=1$ and solving the integral in the R.H.S. of (2.2), with the help of (1.1), we get (2.4).

3. Applications

We derive a number of new results as applications of the theorem.

Case 1. Consider

$$f(xy) = (xy)^{\delta}(1-xy)^{1-\alpha-\beta}(1-x_1xy)^{-\gamma_1}(1-x_2xy)^{-\gamma_2}$$

apply the theorem and use the result [3, p.962(7)] to get

$$\int_0^1\int_0^1\frac{[(1-x)y]^{\alpha}(1-y)^{\beta}(xy)^{\delta}F(X,Y,T(1-x)^{\lambda}y^{\lambda}(1-y)^{\mu}}{(1-x)(1-y)(1-xy)^{\alpha+\beta-1}(1-x_1xy)^{\gamma_1}(1-x_2xy)^{\gamma_2}}\,dxdy$$

$$= \sum_{n=0}^{\infty} c_n \Phi_n(X,Y) T^n B(\alpha+n\lambda, \beta+n\mu) B(\delta+1, n\lambda+n\mu+1)$$

$$\times F_1(\delta+1, \gamma_1\gamma_2, n\lambda+n\mu+\delta+2; x_1, x_2) \tag{2.5}$$

where F_1 is Appell's double hypergeometric series [6, p.53].

When $x_1 = x_2 = 0$, (2.5) reduces to

$$\int_0^1 \int_0^1 \frac{[(1-x)y]^\alpha (1-y)^\beta (xy)^\delta}{(1-x)(1-y)(1-xy)^{\alpha+\beta-1}} F(X,Y,T(1-x)^\lambda y^\lambda (1-y)^\mu) dxdy$$

$$= \sum_{n=0}^{\infty} c_n \Phi_n(X,Y) T^n B(\alpha+n\lambda, \beta+n\mu) B(\delta+1, n\lambda+n\mu+1) \tag{2.6}$$

Further, it is remarked that the result (2.6) is a generalization of the following result:

$$\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^\alpha \left(\frac{1-y}{1-xy}\right)^\beta \frac{(yx)^\rho (1-xy)^{\delta+\alpha+\beta}}{(1-x)(1-y)} \, dx \, dy$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \frac{\Gamma(\rho+1)\,\Gamma(\alpha+\beta+\delta)}{\Gamma(\rho+\alpha+\beta+\delta+1)} \tag{2.7}$$

which for $\rho = 0$ and $\delta = 1-\alpha-\beta$ reduces to (1.5).

Case 2. Consider the generating function[6, p.85] for the Lagrange's polynomials

$$F(X,Y,T) = (1-XT)^{-\gamma}(1-YT)^{-\delta} = \sum_{n=0}^{\infty} g_n^{\gamma,\delta}(X,Y) T^n. \tag{2.8}$$

It follows from the Corollary 2.1 and (2.8) that

$$\int_0^1 \int_0^1 \left[\left(\frac{(1-x)}{1-xy}\right) y\right]^\alpha \left(\frac{1-y}{1-xy}\right)^\beta \frac{(1-xy)}{1-x)(1-y)} (1-TX\theta)^{-\gamma}(1-TY\theta)^{-\delta} \, dx \, dy$$

$$= \sum_{n=0}^{\infty} T^n \, g_n^{\gamma,\delta}(X,Y) \frac{\Gamma(\alpha+n\lambda)\,\Gamma(\beta+n\mu)}{\Gamma(\alpha+\beta+n\lambda+n\mu)} \tag{2.9}$$

where

$$\theta = \left[\frac{(1-x)y}{1-xy}\right]^\lambda \left(\frac{1-y}{1-xy}\right)^\mu. \tag{2.10}$$

(2.9) is a generalization of (1.3) or its equivalent form (1.6).

For $\lambda = \mu = 1$ and $\delta = 0$, (2.9) gives

$$\int_0^1 \int_0^1 \left[\left(\frac{(1-x)}{1-xy} \right) y \right]^\alpha \left(\frac{1-y}{1-xy} \right)^\beta \frac{(1-xy)}{1-x)(1-y)} (1-TX\omega)^{-\gamma} \, dx \, dy$$

$$= B(\alpha, \beta) \, {}_3F_2 \left[\alpha, \beta, \gamma; \frac{\alpha+\beta}{2}, \frac{\alpha+\beta+1}{2}; \frac{TX}{4} \right] \tag{2.11}$$

where $\omega = \dfrac{(1-x)y(1-y)}{(1-xy)^2}$. Note that (2.11) is a correct form of the result recently given by Garg and Gupta [2, p.142(16)].

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