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ON CERTAIN ETA-FUNCTIONS IDENTITIES

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Dedicated to Professor G.E. Andrews on his seventieth birthday

Abstract: In this paper, we establish certain Eta-function identities.

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1. Introduction

For α and q real or complex ($|q| < 1$), we define

$$[\alpha]_n \equiv [\alpha; q]_n = (1-\alpha)(1-\alpha q)\dots(1-\alpha q^{n-1}), \quad n > 0, \quad [\alpha]_0 = 1$$

$$[\alpha]_\infty \equiv [\alpha; q]_\infty = \prod_{n=0}^{\infty}(1-\alpha q^n).$$

With the help of above notations, we define a basic hypergeometric function

$$ {}_r\Phi_s \left[\begin{array}{ll} a_1, a_2, \dots, a_r & ; q; z \\ b_1, b_2, \dots, b_s & ; \end{array} \right] = \sum_{n=0}^{\infty} \frac{[a_1]_n [a_2]_n \dots [a_r]_n z^n}{[q]_n [b_1]_n [b_2]_n \dots [b_s]_n}, \qquad (1.1)$$

valid for $|z| < 1$.

We also define a basic bilateral hypergeometric function

$$ {}_r\Psi_r \left[\begin{array}{ll} a_1, a_2, \cdots, a_r & ; q; z \\ b_1, b_2, \cdots, b_r & ; \end{array} \right] = \sum_{n=-\infty}^{\infty} \frac{[a_1]_n [a_2]_n \dots [a_r]_n z^n}{[b_1]_n [b_2]_n \dots [b_r]_n} \qquad (1.2)$$

valid for $|b_1 b_2 \dots / a_1 a_2 \dots a_r| < |z| < 1$.

(1.2) reduces to (1.1) if any of the denominator parameters tends to q.

We define Dedekind Eta function

$$\eta(\tau)e^{-\pi i\tau/12} = [q]_\infty, \text{ where } q = e^{2\pi i\tau}, \text{ Im}(\tau) > 0.$$

We shall also make use of the following Ramanujan's ${}_1\Psi_1$ summation

$${}_1\Psi_1\left[\begin{array}{ll} a & ;q;z \\ b & ; \end{array}\right] = \frac{[b/a]_\infty\,[az]_\infty\,[q/az]_\infty\,[q]_\infty}{[q/a]_\infty\,[b/az]_\infty\,[b]_\infty\,[z]_\infty} \tag{1.3}$$

Any other notation appearing herein shall carry its usual meaning.

2. Main Results

In this section, we shall discuss our main results.

Setting $a = \frac{1}{\alpha}$, $b = \beta$ and replacing z by $-\alpha z q^{1/2}$ in (1.3), we get

$$\frac{[-zq^{1/2}]_\infty\,[-q^{1/2}/z]_\infty\,[q]_\infty\,[\alpha\beta]_\infty}{[-\alpha zq^{1/2}]_\infty\,[-\beta/zq^{1/2}]_\infty\,[\alpha q]_\infty\,[\beta]_\infty}$$

$$= \sum_{k=1}^{\infty}\frac{[1/\alpha]_k\,(-\alpha zq^{1/2})^k}{[\beta]_k} + \sum_{k=0}^{\infty}\frac{[q/\beta]_k\,(-\beta q^{-1/2})^k\,z^{-k}}{[\alpha q]_k} \tag{2.1}$$

On differentiating both sides of (2.1) with respect to z, we set after some simplification,

$$\frac{[q]_\infty\,[\alpha\beta]_\infty}{[\alpha]_\infty\,[\beta]_\infty}\left\{\frac{[-zq^{1/2}]_\infty\,[-q^{1/2}/z]_\infty}{[-\alpha zq^{1/2}]_\infty\,[-\beta/zq^{1/2}]_\infty}\right\}\left\{\frac{q^{1/2}}{1+zq^{1/2}} + \frac{q^{3/2}}{1+zq^{3/2}} + \ldots\right.$$

$$-\left(\frac{q^{1/2}}{z^2(1+q^{1/2}/z)} + \frac{q^{3/2}}{z^2(1+q^{3/2}/z)} + \ldots\right) - \left(\frac{\alpha q^{1/2}}{1+\alpha zq^{1/2}} + \frac{\alpha q^{3/2}}{1+\alpha zq^{3/2}} + \ldots\right)$$

$$\left. + \left(\frac{\beta q^{-1/2}}{z^2(1+\beta q^{-1/2}/z)} + \frac{\beta q^{1/2}}{z^2(1+\beta q^{1/2}/z)} + \frac{\beta q^{3/2}}{z^2(1+\beta q^{3/2}/z)}\right)\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(k+1)[q/\alpha]_k(-\alpha z)^k\,q^{(k+1)/2}}{[\beta]_{k+1}} - \sum_{k=0}^{\infty}\frac{k[q/\beta]_k\,(-\beta)^k q^{-k/2}\,z^{-k-1}}{[\alpha]_{k+1}} \tag{2.2}$$

Now putting $z = -q^{-1/2}$ in (3.2), Bhargava and Somashekara [1] received the following result,

$$\frac{[q]^3_\infty\,[\alpha\beta]_\infty}{[\alpha]^2_\infty\,[\beta]^2_\infty} = \sum_{k=0}^{\infty}\frac{(k+1)[q/\alpha]_k\alpha^k}{[\beta]_{k+1}} + \sum_{k=0}^{\infty}\frac{k[q/\beta]_k\,\beta^k}{[\alpha]_{k+1}} \tag{2.3}$$

and made use of this relation to derive several Eta-function identities.

In this paper we would like to point out that proper choice of other values of z (2.2) can lead to several new and interesting Eta function identities. We shall also derive several other identities from (2.3) which were possibily not noticed by Bhargava and Somashekara [1].

For $z = -q^{1/2}$ (2.2) leads to the following new relation,

$$\frac{(1-\alpha)}{(1-\beta/q)}\frac{[q]_\infty^3[\alpha\beta]_\infty}{[\alpha]_\infty^2[\beta]_\infty^2} = q\sum_{k=0}^\infty \frac{(k+1)[q/\alpha]_k(\alpha q)^k}{[\beta]_{k+1}} + \sum_{k=0}^\infty \frac{k[q/\beta]_k(\beta/q)^k}{[\alpha]_{k+1}} \quad (2.4)$$

3. Eta-Function Identities

In this section, we shall establish certain interesting Eta-function identities.

(i) Taking $\alpha = w$ and $\beta = w^2 q\,(w = e^{2\pi i/3})$ in (2.3), we get the following identity

$$\frac{1}{(1-w^2)}\frac{\eta^6(\tau)}{\eta^2(3\tau)} = \sum_{k=0}^\infty \frac{(k+1)\,w^k}{(1-w^2q^{k+1})} + \sum_{k=0}^\infty \frac{k\,w^{2k}\,q^k}{(1-wq^k)} \quad (3.1)$$

(ii) Next, setting $\alpha = -w$ and $\beta = -w^2 q$ in (2.3), we get,

$$\frac{\eta^2(\tau)\,\eta^2(2\tau)\,\eta^2(3\tau)}{(1+w^2)\,\eta^2(6\tau)} = \sum_{k=0}^\infty \frac{(k+1)(-w)^k}{(1+w^2q^{k+1})} + \sum_{k=0}^\infty \frac{k(-w^2q)^k}{(1+wq^k)} \quad (3.2)$$

(iii) Again, taking $\alpha = iq$ and $\beta = -iq$ in (2.3), we get

$$\frac{1}{(1-q)}\frac{\eta^4(\tau)\,\eta^2(2\tau)}{\eta^2(4\tau)} = \sum_{k=0}^\infty \frac{(k+1)(1+i)(iq)^k}{(1+iq^{k+1})} + \sum_{k=0}^\infty \frac{k(1-i)(-iq)^k}{(1-iq^{k+1})} \quad (3.3)$$

(iv) Further, setting $\alpha = iq$ and $\beta = -iq$ in (2.3), we get,

$$\frac{q^{-3/8}\,\eta^3(\tau)\,\eta^3(2\tau)\,\eta^2(8\tau)}{\eta^4(4\tau)} = \sum_{k=0}^\infty \frac{(k+1)(iq)^k}{(1+iq^{2k+1})} + \sum_{k=0}^\infty \frac{k(-iq)^k}{(1-iq^{2k+1})} \quad (3.4)$$

(v) setting $\alpha = \beta = q$ in (2.3), we get,

$$\frac{\eta^8(2\tau)}{\eta^4(\tau)} = \sum_{k=0}^\infty \frac{(k+1)\,q^{(2k+1)/2}}{(1-q^{2k+1})} + \sum_{k=0}^\infty \frac{k\,q^{(2k+1)/2}}{(1-q^{2k+1})} \quad (3.5)$$

Now, we shall establish Eta function identities with the help of (2.4). Replacing q by q^2 in (2.4), we get,

$$\frac{(1-\alpha)}{(1-\beta/q^2)}\frac{[q^2;q^2]_\infty^3\,[\alpha\beta;q^2]_\infty}{[\alpha;q^2]_\infty^2\,[\beta;q^2]_\infty^2} = q^2\sum_{k=0}^{\infty}\frac{(k+1)\,[q^2/\alpha;q^2]_k\,(\alpha q^2)^k}{[\beta;q^2]_{k+1}} + \sum_{k=0}^{\infty}\frac{k\,[q^2/\beta;q^2]_k\,(\beta/q^2)^k}{[\alpha;q^2]_{k+1}} \tag{3.6}$$

Now, replacing β by βq in (2.4) and β by βq^2 in (3.6), we get, respectively, the following relations,

$$\frac{(1-\alpha)}{(1-\beta)}\frac{[q;q]_\infty^3\,[\alpha\beta q;q]_\infty}{[\alpha;q]_\infty^2\,[\beta q;q]_\infty^2} = q\sum_{k=0}^{\infty}\frac{(k+1)\,[q/\alpha;q]_k\,(\alpha q)^k}{[\beta q;q]_{k+1}} + \sum_{k=0}^{\infty}\frac{k\,[q/\beta;q]_k\,\beta^k}{[\alpha;q]_{k+1}} \tag{3.7}$$

and

$$\frac{(1-\alpha)}{(1-\beta)}\frac{[q^2;q^2]_\infty^3\,[\alpha\beta q^2;q^2]_\infty}{[\alpha;q^2]_\infty^2\,[\beta q^2;q^2]_\infty^2} = q^2\sum_{k=0}^{\infty}\frac{(k+1)\,[q^2/\alpha;q^2]_k\,(\alpha q^2)^k}{[\beta q^2;q^2]_{k+1}} + \sum_{k=0}^{\infty}\frac{k\,[1/\beta;q^2]_k\,\beta^k}{[\alpha;q^2]_{k+1}} \tag{3.8}$$

(vi) If we replace α and β by wq and w^2q in (3.8), we get, after some simplification, the following interesting identity,

$$\frac{q^{-1/2}(1+q+q^2)}{(1-q^2)}\frac{\eta^2(\tau)\,\eta^2(2\tau)\,\eta^2(6\tau)}{\eta^2(3\tau)}$$

$$= q^2(1-w^2q)\sum_{k=0}^{\infty}\frac{(k+1)(wq^3)^k}{(1-w^2q^{2k+3})} + (1-w/q)\sum_{k=0}^{\infty}\frac{k\,(w^2q)^k}{(1-wq^{2k+1})},\quad w = e^{2\pi i/3} \tag{3.9}$$

(vii) Next, we put $\alpha = w$ and $\beta = w^2q$ in (3.7), we get, after some simplification,

$$\frac{(1-w^2q)}{(1-q)(1-w)}\frac{\eta^6(\tau)}{\eta^2(3\tau)} = q(1-w^2q)\sum_{k=0}^{\infty}\frac{(k+1)(wq)^k}{(1-w^2q^{k+2})} + (1-w/q)\sum_{k=0}^{\infty}\frac{k\,(w^2q)^k}{(1-wq^k)} \tag{3.10}$$

(viii) Again, if we take $\alpha = -w_1$, $\beta = -w^2 q$ in (3.7), we get, after some simplification,

$$\frac{(1+w^q)}{(1-q)(1+w)}\,\frac{\eta^2(\tau)\,\eta^2(2\tau)\,\eta^2(3\tau)}{\eta^2(6\tau)} = q(1+w^2q)\sum_{k=0}^{\infty}\frac{(k+1)(-wq)^k}{(1+w^2q^{k+2})} + (1+w/q)\sum_{k=0}^{\infty}\frac{k\,(-w^2q)^k}{(1+wq^k)} \tag{3.11}$$

(ix) Further, taking $\alpha = \beta = q$ in (3.8), we get,

$$\frac{q^{1/2}\,\eta^8(2\tau)}{(1+q)\,\eta^4(\tau)} \;=\; q^3\sum_{k=0}^{\infty}\frac{(k+1)q^{3k}}{(1-q^{2k+3})} - \sum_{k=0}^{\infty}\frac{k\,q^k}{(1-q^{2k+1})} \tag{3.12}$$

(x) Next, if we put $\alpha = \beta = -q$ in (3.8), we get,

$$\frac{q^{-1/2}\,\eta^4(\tau)\,\eta^4(4\tau)}{\eta^4(2\tau)} = q^2(1-q)\sum_{k=0}^{\infty}\frac{(k+1)(-)^k\,q^{3k}}{(1+q^{3k+3})} + \left(\frac{1-q}{q}\right)\sum_{k=0}^{\infty}\frac{k\,(-)^k\,q^k}{(1+q^{2k+1})} \tag{3.13}$$

(xi) Again taking $\alpha = -wq$ and $\beta = -w^2 q$ in (3.8), we get,

$$\frac{q^{-1/2}(1-q+q^2)}{(1-q^2)}\,\frac{\eta^8(2\tau)\,\eta^2(3\tau)\,\eta^2(12\tau)}{\eta^4(6\tau)\,\eta^2(4\tau)\,\eta^2(\tau)} \;=\; q^2(1+w^2q)\sum_{k=0}^{\infty}\frac{(k+1)(-)^k(wq^3)^k}{(1+w^2q^{2k+3})} + (1+w/q)\sum_{k=0}^{\infty}\frac{k\,(-)^k\,(w^2q)^k}{(1+wq^{2k+1})}, \tag{3.14}$$

(xii) Lastly, if we put $\alpha = iq$ and $\beta = -iq$ in (3.8), we get,

$$\frac{q^{-1/6}\,\eta^6(2\tau)}{(1-q^2)\,\eta^2(4\tau)} = q^2(1+iq)\sum_{k=0}^{\infty}\frac{(k+1)(iq)^k}{(1+iq^{2k+3})} + \left(1-\frac{i}{q}\right)\sum_{k=0}^{\infty}\frac{k\,(-iq)^k}{(1-iq^{2k+1})} \tag{3.15}$$

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