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On certain q-series identities

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Abstract: In this paper, an attempt has been made to establish certain q-series identities which are different from Rogers-Rananujan type identities.
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1. Introductions Notations and Definitions:

Throughout this note, we shall adopt following definitions and notations. The q-shifted factorial is defined by,

$$[a;q]_0 = 1, \quad [a;q]_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad n = 1, 2, 3, ...$$

and

$$[a;q]_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

For products of q-shifted factorials, we use the short notation,

$$[a_1, a_2, a_3, ..., a_r; q]_n = [a_1;q]_n[a_2;q]_n...[a_r;q]_n,$$

where n is an integer or infinity. Basic hypergeometric series is defined by,

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s; q^\lambda \end{array}\right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n q^{\lambda n(n-1)/2}}{[q, b_1, b_2, ..., b_s; q]_n},$$

which is convergent in the whole complex plane if $\lambda \neq 0$ and for $\lambda = 0$, it converges for max. $(|q|, |z|) < 1$, provided $r = s + 1$.

Rogers and Ramanujan [6] published a paper containing following identities,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{[q;q]_n} = \frac{1}{[q;q^5]_\infty [q^4;q^5]_\infty} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{[q;q]_n} = \frac{1}{[q^2;q^5]_\infty [q^3;q^5]_\infty} \tag{1.2}$$

After the publication of this paper, many mathematicians notably, Bailey [2], Slater [7], Verma, A. [9], Denis [3], Singh, V.N. [8] attempted and also succeeded in establishing Rogers-Ramanujan type identities. There are two types of identities available in the literature.

(i) Rogers-Ramanujan type identities

(ii) Identities different from Rogers-Ramanujan type.

An identity in which a series is expressed in the form of product like (1.1) and (1.2) is called Rogers-Ramanujan type identity. On the other hand if a series is represented by an equivalent another series then it is called an identity different from Rogers-Ramanujan type. The main aim of this paper is to establish identities of second kind.

2. Main Results

In the 'lost' notebook of Ramanujan [5], he has mentioned following beautiful theorems.

Theorem 1. If $a, b \neq -q^{-n}$, then

$$\rho(a,b) - \rho(b,a) = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{[q, aq/b, bq/a; q]_\infty}{[-aq, -bq; q]_\infty},$$

where

$$\rho(a,b) = \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2} (a/b)^n}{[-aq;q]_n}. \tag{2.1}$$

Let us consider the Rogers-Fine identity, viz.,

$$\sum_{n=0}^{\infty} \frac{[\alpha;q]_n}{[\beta;q]_n} z^n = \sum_{n=0}^{\infty} \frac{[\alpha, \alpha zq/\beta; q]_n (\beta z)^n (1 - \alpha z q^{2n}) q^{n(n-1)}}{[\beta;q]_n [z;q]_{n+1}} \tag{2.2}$$

Replacing z by z/α and then taking $\alpha \to \infty$ in (2.2) we have

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n-1)/2} z^n}{[\beta;q]_n} = \sum_{n=0}^{\infty} \frac{(-)^n q^{3n(n-1)/2} [zq/\beta;q]_n (\beta z)^n (1 - zq^{2n})}{[\beta;q]_n} \tag{2.3}$$

Again, taking $\beta = -aq$ and $z = aq/b$ in (2.3) we find

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}(a/b)^n}{[-aq;q]_n} = \sum_{n=0}^{\infty} \frac{[-q/b;q]_n (a^2/b)^n (1-aq^{2n+1}/b) q^{n(3n+1)/2}}{[-aq;q]_n}$$

So,

$$\rho(a,b) = \left(1+\frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}(a/b)^n}{[-aq;q]_n}$$

$$= \left(1+\frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{[-q/b;q]_n (a^2/b)^n (1-aq^{2n+1}/b) q^{n(3n+1)/2}}{[-aq;q]_n} \tag{2.4}$$

From (2.4) we find a general identity,

$$\sum_{n=0}^{\infty} \frac{(-)^n (a/b)^n q^{n(n+1)/2}}{[-aq;q]_n} = \sum_{n=0}^{\infty} \frac{[-q/b;q]_n (1-aq^{2n+1}/b)(a^2/b)^n q^{n(3n+1)/2}}{[-aq;q]_n} \tag{2.5}$$

(i) Taking a=b=1 in (2.5) we get,

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{[-q;q]_n} = \sum_{n=0}^{\infty} (1-q^{2n+1}) q^{n(3n+1)/2} = 1 - \sum_{n=1}^{\infty} q^{n(3n-1)/2}(1-q^n) \tag{2.6}$$

(ii) Taking a=b=-1 in (2.5) we get,

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{[q;q]_n} = \sum_{n=0}^{\infty} (1-q^{2n+1})(-)^n q^{n(3n+1)/2}$$

$$= \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n+1)/2} = [q;q]_\infty$$

Thus we find,

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{[q;q]_n} = [q;q]_\infty. \tag{2.7}$$

(iii) Taking a=-b=1 in (2.5) we get,

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{[-q;q]_n} = \sum_{n=0}^{\infty} \frac{[q;q]_n (1+q^{2n+1})(-)^n q^{n(3n+1)/2}}{[-q;q]_n}$$

(iv) Taking -a=b=1 in (2.5) we have,

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{[q;q]_n} = \sum_{n=0}^{\infty} \frac{[-q;q]_n(1+q^{2n+1})q^{n(3n+1)/2}}{[q;q]_n} \tag{2.8}$$

(v) Taking a=1, b=q in (2.5) we get,

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n-1)/2}}{[-q;q]_n} = 2\sum_{n=1}^{\infty} \frac{(1-q^n)q^{n(3n-1)/2}}{[-q;q]_n} \tag{2.9}$$

(vi) Taking a=-1, b=q in (2.5) we have,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[q;q]_n} = \sum_{n=0}^{\infty} \frac{[-1;q]_n(1+q^{2n})q^{n(3n-1)/2}}{[q;q]_n} \tag{2.10}$$

(vii) $a = 1/q$ and b=1 in (2.5) we find,

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n-1)/2}}{[-1;q]_n} = \frac{1}{2}\sum_{n=0}^{\infty}(1+q^n)(1-q^{2n})q^{3n(n-1)/2}$$

(viii) a=1/q and b=-1 in (2.5) we get,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[-1;q]_n} = \sum_{n=0}^{\infty} \frac{[q;q]_n(1-q^{2n})(-)^n q^{3n(n-1)/2}}{[-1;q]_n} \tag{2.11}$$

(ix) Taking a=1/q and b=q in (2.5) we have

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n-3)/2}}{[-1;q]_n} = \sum_{n=0}^{\infty}(1-q^{2n-1})q^{n(3n-5)/2}. \tag{2.12}$$

(x) Taking a =1/q and b=-q in (2.5) we have,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-3)/2}}{[-1;q]_n} = 1. \tag{2.13}$$

(xi) Taking $a = \omega$ and $b = \omega^2$ in (2.5) we get,

$$\sum_{n=0}^{\infty} \frac{(-)^n \omega^{2n} q^{n(n+1)/2}}{[-\omega q;q]_n} = \sum_{n=0}^{\infty}(1-\omega^2 q^{2n+1})q^{n(3n+1)/2} \tag{2.14}$$

(xii) Taking $a=\omega^2$ and $b=\omega$ in (2.5) we get,

$$\sum_{n=0}^{\infty}\frac{(-)^n\omega^n q^{n(n+1)/2}}{[-\omega^2 q;q]_n}=\sum_{n=0}^{\infty}(1-\omega q^{2n+1})q^{n(3n+1)/2} \tag{2.15}$$

(xiii) Taking $a=i$ and $b=-i$ in (2.5) we have

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{[-iq;q]_n}=\sum_{n=0}^{\infty}(1+q^{2n+1})q^{n(3n+1)/2}(-)^{3n/2}. \tag{2.16}$$

(xiv) Taking $a=i$ and $b=i$ in (2.5) we have

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)/2}}{[-iq;q]_n}=\sum_{n=0}^{\infty}\frac{[iq;q]_n(1-q^{2n+1})q^{n(3n+1)/2}(-)^{n/2}}{[-iq;q]_n}. \tag{2.17}$$

(xv) Taking $a=-i$, $b=i$ gives

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{[iq;q]_n}=\sum_{n=0}^{\infty}(1+q^{2n+1})(-)^{n/2}q^{n(3n+1)/2} \tag{2.18}$$

Let us consider the following result,

$$\frac{(1-\alpha)x}{(1-\gamma)}\sum_{n=0}^{\infty}\frac{(\alpha q;q)_n(-)^n x^n}{[\gamma q;q]_n}$$

$$=\frac{(1-\alpha)x}{(1-\gamma)+}\,\frac{x(1-\alpha q)}{1+}\,\frac{(1-q)(\gamma+\alpha xq)}{(1-\gamma)+}$$

$$\frac{xq(1-\alpha q^2)}{1+}\,\frac{(1-q^2)(\gamma+\alpha xq^2)}{(1-\gamma)+...}$$

[Agarwal 1;(4.3) p.84]

Taking x/α for x and then $\alpha\to\infty$ gives

$$\frac{-x}{(1-\gamma)}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}x^n}{[\gamma q;q]_n}=\frac{-x}{(1-\gamma)+}\,\frac{-xq}{1+}$$

$$\frac{(1-q)(\gamma+xq)}{(1-\gamma)+}\,\frac{-xq^3}{1+}\,\frac{(1-q^2)(\gamma+xq^2)}{(1-\gamma)+...}$$

Taking $\gamma = -a$ and $x = -a/b$ in it we have

$$\frac{a/b}{(1+a)}\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)/2}(a/b)^n}{[-aq;q]_n} = \frac{a/b}{(1+a)+}\,\frac{aq/b}{1+}$$

$$\frac{(1-q)(-a-aq/b)}{(1+a)+}\,\frac{aq^3/b}{1+}\,\frac{(1-q^2)(-a-aq^2/b)}{(1+a)+\ldots}$$

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)/2}(a/b)^n}{(-aq;q)_n} = \frac{1+a}{(1+a)+}\,\frac{aq/b}{1-}$$

$$\frac{a(1-q)(1+q/b)}{(1+a)+}\,\frac{aq^3/b}{1-}\,\frac{a(1-q^2)(1-q^2/b)}{(1+a)+\ldots}$$

Thus we have

$$\rho(a,b) = \frac{(1+a)(1+1/b)}{(1+a)+}\,\frac{aq/b}{1-}\,\frac{a(1-q)(1+q/b)}{(1+a)+}\,\frac{aq^3/b}{1-\ldots} \tag{2.19}$$

$$= \frac{(1+a)(1+b)}{b(1+a)+}\,\frac{aq}{1-}\,\frac{a(1-q)(b+q)}{b(1+a)+}\,\frac{aq^3}{1-\ldots} \tag{2.20}$$

$$\rho(1,1) = \frac{4}{2+}\,\frac{q}{1-}\,\frac{(1-q)(1+q)}{2+}\,\frac{q^3}{1-}\,\frac{1-q^4}{2+\ldots}$$

$$= \frac{4}{2+}\,\frac{q}{1-}\,\frac{1-q^2}{2+}\,\frac{q^3}{1-}\,\frac{1-q^4}{2+}\,\frac{q^5}{1-\ldots} \tag{2.21}$$

$$\rho(1,q) = \frac{2(1+q)}{2q+}\,\frac{q}{1-}\,\frac{2q(1-q)}{2q+}\,\frac{q^3}{1-}\,\frac{2q^2(1-q^2)}{2q+}\,\frac{q^5}{1-\ldots} \tag{2.22}$$

$$\rho(q,q) = \frac{(1+q)^2}{q(1+q)+}\,\frac{q^2}{1-}\,\frac{2q^2(1-q)}{q(1+q)+}\,\frac{q^4}{1-}\,\frac{q^2(1-q^2)(1+q)}{q(1+q)+}\,\frac{q^6}{1-\ldots} \tag{2.23}$$

$$\rho(q,1) = \frac{2(1+q)}{(1+q)+}\,\frac{q^2}{1-}\,\frac{q(1-q^2)}{(1+q)+}\,\frac{q^4}{1-}\,\frac{q(1-q^4)}{(1+q)+}\,\frac{q^6}{1-\ldots} \tag{2.24}$$

$$\rho(\omega,\omega) = \frac{\omega}{-1+}\,\frac{\omega q}{1-}\,\frac{\omega(1-q)(\omega+q)}{-1+}\,\frac{\omega q^3}{1-}\,\frac{\omega(1-q^2)(\omega+q^2)}{-1+}\,\frac{\omega q^5}{1-\ldots} \tag{2.25}$$

3. Ramanujan's Second Theorem

If $\rho(a,b,c) = \rho(a,b,c;q)$

$$= \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{[c;q]_n(a/b)^n(-)^n q^{n(n+1)/2}}{[-aq;q]_n[-c/b;q]_{n+1}}$$

$$\rho(a,b,c)-\rho(a,b,c;q)$$

$$=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{[c,aq/b,bq/a,q;q]_\infty}{[-c/a,-c/b,-aq,-bq;q]_\infty} \tag{3.1}$$

Let us consider the following transformation [Gasper-Rahman 4;App. III 9,10]

$${}_3\Phi_2\left[\begin{matrix}a,b,c;q;de/abc\\ d,e\end{matrix}\right]=\frac{[e/a,de/bc;q]_\infty}{[e,de/abc;q]_\infty}\,{}_3\Phi_2\left[\begin{matrix}a,b,c;q;de/abc\\ d,e\end{matrix}\right]$$

$$=\frac{[b,de/ab,de/bc;q]_\infty}{[d,e,de/abc;q]_\infty}\,{}_3\Phi_2\left[\begin{matrix}d/b,e/b,de/abc;q;b\\ de/ab,de/bc\end{matrix}\right]. \tag{3.2}$$

[Gasper-Rahman 4; App. III (9,10)]

Taking $a\to\infty, b=q$ and then replacing d by -aq and e by -cq/b in (3.2)

$$\rho(a,b,c;q)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-)^n(c)_nq^{n(n+1)/2}(a/b)^n}{[-aq;q]_n[-c/b;q]_{n+1}}$$

$$=\left(1+\frac{1}{b}\right)\frac{[aq/b;q]_\infty}{[-c/b;q]_\infty}\sum_{n=0}^{\infty}\frac{[-a,-aq/c;q]_n(c/b)^nq^{n(n+1)/2}}{[q,-aq,aq/b;q]_n}$$

$$=\left(1+\frac{1}{b}\right)\frac{[q,aq/b;q]_\infty}{[-aq,-c/b;q]_\infty}\sum_{n=0}^{\infty}\frac{[-a,-c/b;q]_nq^n}{[q,aq/b;q]_n}. \tag{3.3}$$

For a=1 we have

$$\sum_{n=0}^{\infty}\frac{[b,c;q]_n\left(\frac{de}{bc}\right)^n}{[dq,e;q]_n}=\frac{1}{1-}\;\frac{\frac{de}{bc}(1-d)(1-b)(1-c)/(1-e)(1-d)(1-dq)}{(1-e/q)/(1-e)+}$$

$$\frac{\frac{e}{q}(1-q)(1-dq/b)(1-dq/c)/(1-e)(1-dq)(1-dq^2)}{1-}$$

$$\frac{\frac{deq}{bc}(1-dq)(1-bq)(1-cq)/(1-eq)(1-dq^2)(1-dq^3)}{(1-e/q)/(1-eq)+}$$

$$\frac{\frac{e}{q}(1-q^2)(1-dq^2/b)(1-dq^2/c)/(1-eq)(1-dq^3)(1-dq^4)}{1-}$$

$$\cfrac{\frac{deq^2}{bc}(1-dq^2)(1-bq^2)(1-cq^2)/(1-eq^2)(1-dq^4)(1-dq^5)}{(1-e/q)/(1-eq^2)+\ldots} \tag{3.4}$$

$$= \frac{1}{1-}\,\cfrac{\frac{de}{bc}(1-d)(1-b)(1-c)/(1-d)(1-dq)}{(1-e/q)+}$$

$$\cfrac{\frac{e}{q}(1-q)(1-dq/b)(1-dq/c)/(1-dq)(1-dq^2)}{1-}$$

$$\cfrac{\frac{deq}{bc}(1-dq)(1-bq)(1-cq)/(1-dq^2)(1-dq^3)}{(1-e/q)+}$$

$$\cfrac{\frac{e}{q}(1-q^2)(1-dq^2/b)(1-dq^2/c)/(1-dq^3)(1-dq^4)}{1-}$$

$$\cfrac{\frac{deq^2}{bc}(1-dq^2)(1-bq^2)(1-cq^2)/(1-dq^4)(1-dq^5)}{(1-e/q)+\ldots} \tag{3.5}$$

As $b \to \infty$

$$\sum_{n=0}^{\infty} \frac{[c;q]_n(-)^n q^{n(n-1)/2}(de/c)^n}{[dq,e;q]_n} = \frac{1}{1+}\,\cfrac{\frac{de}{c}(1-d)(1-e)/(1-d)(1-dq)}{(1-e/q)+}$$

$$\cfrac{\frac{e}{q}(1-q)(1-dq/c)/(1-dq)(1-dq^2)}{1+}$$

$$\cfrac{\frac{deq^2}{c}(1-dq)(1-eq)/(1-dq^2)(1-dq^3)}{(1-e/q)+}$$

$$\cfrac{\frac{e}{q}(1-q^2)(1-dq^2/c)/(1-dq^3)(1-dq^4)}{1+}$$

$$\cfrac{\frac{deq^4}{c}(1-dq^2)(1-eq^2)/(1-dq^4)(1-dq^5)}{(1-e/q)+\ldots} \tag{3.6}$$

Taking d=-a, e=-cq/b in it we get

$$\sum_{n=0}^{\infty}\frac{[c;q]_n(-)^n q^{n(n+1)/2}(a/b)^n}{[-aq,-qc/b;q]_n}=\frac{(1+c/b)}{(1+1/b)}\rho(a,b,c;q)$$

$$=\frac{1}{1+}\frac{\frac{aq}{b}(1+a)(1-c)/(1+a)(1+aq)}{(1+c/b)-}$$
$$\frac{\frac{c}{b}(1-q)(1+aq/c)/(1+aq)(1+aq^2)}{1-}$$
$$\frac{\frac{aq^3}{b}(1+aq)(1-cq)/(1+aq^2)(1+aq^3)}{(1+c/b)-}$$
$$\frac{\frac{c}{b}(1-q^2)(1+aq^2/c)/(1+aq^3)(1+aq^4)}{1-}$$
$$\frac{\frac{aq^5}{b}(1+aq^2)(1-cq^2)/(1+aq^4)(1+aq^5)}{(1+c/b)-...} \quad (3.7)$$

A number of interesting identities can also be scored.

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