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# GENERATING RELATIONS OF GAUSS HYPERGEOMETRIC FUNCTION

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**Abstract:** By making use of familiar Laplace and inverse Laplace transform technique, we obtain generating functions involving Gauss and Kampé de Fériet hypergeometric functions. Certain special cases are also considered.

**Keywords and Phrases:** Generalized hypergeometric functions, Kampé de Fériet hypergeometric functions
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## 1. Introduction

Let $(\lambda)_n$ denote the Pochhamer symbol

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n=0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (n \in N). \end{cases} \tag{1.1}$$

and ${}_pF_q$ denote the generalized hypergeometric function

$${}_pF_q\left[\begin{array}{ll} a_1,\cdots,a_p & ; \\ & z \\ b_1,\cdots,b_q & ; \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} \tag{1.2}$$

with $p$ numerator and $q$ denominator parameters.

By Kampé de Fériet's hypergeometric functions of two variables, we mean

$$F^{p:r;u}_{q:s;v}\left[\begin{array}{llll} \alpha_1,\cdots,\alpha_p & :\rho_1,\cdots,\rho_r & ;\lambda_1,\cdots,\lambda_u & ; \\ & & & x,y \\ \beta_1,\cdots,\beta_q & :\sigma_1,\cdots,\sigma_s & ;\mu_1,\cdots,\mu_v & ; \end{array}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_{m+n}\cdots(\alpha_p)_{m+n}\,(\rho_1)_m\cdots(\rho_r)_m\,(\lambda_1)_n\cdots(\lambda_u)_n}{(\beta_1)_{m+n}\cdots(\beta_q)_{m+n}\,(\sigma_1)_m\cdots(\sigma_s)_m\,(\mu_1)_n\cdots(\mu_v)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.3}$$

where $p+r<q+s+1$, $p+u<q+v+1$, $|x|<\infty$, $|y|<\infty$ and $p+r=q+s+1$, $p+u=q+v+1$,

$$\begin{cases} |x|^{1/(p-q)}+|y|^{1/(p-q)}<1, & \text{if } p>q, \\ \max\{|x|,\ |y|\}<1, & \text{if } p\leq q. \end{cases} \tag{1.4}$$

The purpose of this paper is to obtain the following generating functions by using results of Bromberg ([1]; p.15(2.1) and (2.2)) as a main working tools

$$\sum_{r=0}^{\infty}\frac{(\rho)_r\,(\alpha)_r\,(d_D)_r\,z^r}{(\beta)_r\,(e_E)_r\,r!}\,{}_1F_1[1-\rho;\,1-\rho-r;\,t]=e^t\,F^{D+1:1;0}_{E+1:0;0}\left[\begin{array}{ll} \alpha,\,(d_D):\ \rho;\_\ \ ; & \\ & z,-zt \\ \beta,\,(e_E):\ \_;\_\ \ ; & \end{array}\right] \tag{1.5}$$

$$F^{D+1:1;1}_{E+1:0;0}\left[\begin{array}{ll} \alpha,\,(d_D):\ \lambda;\rho\ \ ; & \\ & zt,z \\ \beta,\,(e_E):\ \_;\_\ \ ; & \end{array}\right] = F^{D+2:1;0}_{E+1:1;0}\left[\begin{array}{llll} \alpha,\,\rho+\lambda,\,(d_D) & :\ \lambda & ;\_\ ; & \\ & & & (t-1)z,z \\ \beta,\,(e_E) & :\rho+\lambda & ;\_\ ; & \end{array}\right] \tag{1.6}$$

where for the brevity $(d_D)$ denotes the array of $D$ parameters $d_1,d_2,\cdots,d_D$ with similar interpretation for $(e_E)$.

## 2. Generating Functions

We have ([1], p.15(2.1) and (2.2))

$$\sum_{n=0}^{\infty}{}_2F_1[\rho-n,\alpha;\,\beta;\,z]\,\frac{t^n}{n!}\ =\ e^t\Phi_1[\alpha,\rho;\,\beta;\,z-zt]\qquad |z|<1,\quad |zt|<\infty \tag{2.1}$$

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n\,(\lambda)_n}{(\beta)_n\,n!}\,{}_2F_1[\alpha+n,\,\rho;\,\beta+n;\,z]\ z^nt^n$$
$$=\ F^{2:1;0}_{1:1;0}\left[\begin{array}{llll} \alpha,\,\rho+\lambda & :\lambda & ;\_\ ; & \\ & & & -(1-t)z,z \\ \beta & :\rho+\lambda & ;\_\ ; & \end{array}\right],\qquad |t|<|z|^{-1}. \tag{2.2}$$

On replacing $z$ by $zu$ in (2.1), multiplying both sides $u^{\lambda-1}$ and taking Laplace transform, we get

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^{\infty}{}_2F_1[\rho-n,\alpha;\beta;zu]u^{\lambda-1}e^{-pu}\,du=e^t\int_0^{\infty}u^{\lambda-1}\,e^{-pu}\,\Phi_1[\alpha,\rho;\beta;zu-ztu]du. \tag{2.3}$$

Again expanding ${}_2F_1$ and $\Phi_1$ in series integrating term by term and using $\Gamma(\alpha - n) = \dfrac{(-1)^n\,\Gamma(\alpha)}{(1-\alpha)_n}$, we get

$$\sum_{r=0}^{\infty} \frac{(\rho)_r\,(\alpha)_r\,(\lambda)_r\,z^r}{(\beta)_r\,r!\,p^r}\,{}_1F_1[1-\rho;1-\rho-r;t] = e^t\,F^{2:1;0}_{1:0;0}\left[\begin{array}{lll} \alpha,\,\lambda & :\rho;\_ & ; \\ & & \dfrac{z}{p},-\dfrac{zt}{p} \\ \beta & :\_;\_ & ; \end{array}\right]. \tag{2.4}$$

On the other hand, if we take inverse Laplace transform of (2.4), we get

$$\sum_{r=0}^{\infty} \frac{(\rho)_r\,(\alpha)_r\,(\lambda)_r\,z^r}{(\beta)_r\,(\mu)_r\,r!\,p^r}\,{}_1F_1[1-\rho;1-\rho-r;t] = e^t\,F^{2:1;0}_{2:0;0}\left[\begin{array}{lll} \alpha,\,\lambda : \rho;\_ & ; \\ & \dfrac{z}{p},-\dfrac{zt}{p} \\ \beta,\,u : \_;\_ & ; \end{array}\right]. \tag{2.5}$$

Continuing this process it is not difficult to obtain (1.5) by induction.

Similarly, applying the same Laplace transform technique to (2.2), we get the transformation

$$F^{2:1;1}_{1:0;0}\left[\begin{array}{lll} \sigma,\,\alpha & :\lambda;\rho & ; \\ & & \dfrac{zt}{p},\dfrac{z}{p} \\ \beta & :\_;\_ & ; \end{array}\right] = F^{3:1;0}_{1:1;0}\left[\begin{array}{llll} \alpha,\,\rho+\lambda,\,\sigma & :\lambda & ;\_ & ; \\ & & & \dfrac{(t-1)z}{p},\dfrac{z}{p} \\ \beta & :\rho+\lambda & ;\_ & ; \end{array}\right]. \tag{2.6}$$

Now starting again from (2.2) and making use of Laplace and inverse Laplace transform, we can prove (1.6) by induction.

## 3. Special Cases

For $t = 1$ and $\alpha = \beta$ (1.6) reduces to a known result ([10]; p.28(32))

$$F^{D:1;1}_{E:0;0}\left[\begin{array}{ll} (d_D) : \lambda;\rho & ; \\ & z,z \\ (e_E) : \_;\_ & ; \end{array}\right] = {}_{D+1}F_E[(d_D),\,\lambda+\rho;\,(e_E);\,z] \tag{3.1}$$

whereas for $t = 0$ it yields another known result ([10]; p.29(45))

$$F^{D:1;0}_{E:1;0}\left[\begin{array}{ll} (d_D) : \lambda;\_ & ; \\ & -z,z \\ (e_E) : \rho;\_ & ; \end{array}\right] = {}_{D+1}F_{E+1}[(d_D),\,\rho-\lambda;\,(e_E),\rho;\,z] \tag{3.2}$$

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