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# A PAIR OF UNSYMMETRRICAL FOURIER KERNELS INVOLVING *I*-FUNCTIONS

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**Abstract:** In the present paper an attempt has been made to show that $I$-function and a finite sum of $H$-functions form a pair of unsymmetrical Fourier kernels under a specified set of conditions. The reciprocity has been established by the method of mean square convergence. In addition, a set of sufficient conditions for uniform convergence of the $I$-function has also been obtained as a concomitant result. Further, result obtained by Kesarwani [6,7] for unsymmetrical Fourier kernels follow as special cases.

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## 1. Introduction

The functions $K_1(x)$ and $K_2(x)$ are said to form a pair of Fourier kernels, if the reciprocal equations

$$f(x) \;=\; \int_0^\infty K_1(xy)g(y)\,dy \tag{1.1}$$

and

$$g(x) \;=\; \int_0^\infty K_2(xy)f(y)\,dy \tag{1.2}$$

are simultaneously satisfied. The kernels are said to be symmetrical if $K_1(x) = K_2(x)$ and unsymmetrical if $K_1(x) \neq K_2(x)$. The Fourier kernels satisfying equations (1.1) and (1.2) have been obtained from time to time by various researchers. Fox [1] obtained $G$ and $H$-functions under some restrictions as symmetric Fourier kernels and extending his work, Kesarwani [5-7], Gupta [2] and Saxena [9] established the certain $G$ and $H$-functions as unsymmetrical Fourier kernels.

In the present paper we shall show that $H(x)$ and $K(x)$ are reciprocal kernels, where $H(x)$ is defined as finite sum of $H$-functions by $H(x) = \sum_{i=1}^{r} H_i(x)$ such as

$$\begin{aligned} H_i(x) &= H_{p_i,q_i}^{q_i-m,p_i-n}[x] \\ &= \frac{1}{2\pi\omega} \int_L \frac{\prod_{j=m+1}^{q_i} \Gamma(b_{ji} - \beta_{ji}s) \prod_{j=n+1}^{p_i} \Gamma(1 - a_{ji} + \alpha_{ji}s)}{\prod_{j=1}^{m} \Gamma(1 - b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(a_j - \alpha_j s)} x^s \, ds. \end{aligned} \tag{1.3}$$

Here $\omega$ is square root of $-1$ and the conditions for the convergence of $H$-functions are as quoted in literature [1].

Again definition of $K(x)$ involves $I$-function which is enunciated by Saxena [9] as follows:

$$\begin{aligned} I[x] = I\left[\, x \,\middle|\, \begin{matrix} (a_j, \alpha_j), (a_{ji}, \alpha_{ji}) \\ \\ (b_j, \beta_j), (b_{ji}, \beta_{ji}) \end{matrix} \right] &= I_{p_i,q_i:r}^{m,n}\left[\, x \,\middle|\, \begin{matrix} \{(a_j, \alpha_j)_{1,n}\}, \{(a_{ji}, \alpha_{ji})_{n+1,p_i})\} \\ \\ \{(b_j, \beta_j)_{1,m}\}, \{(b_{ji}, \beta_{ji})_{m+1,q_i}\} \end{matrix} \right] \\ &= \frac{1}{2\pi\omega} \int_L \theta(s) x^s \, ds, \end{aligned} \tag{1.4}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right\}}$$

with all other conditions as already detailed in [4,10].

We take

$$K(x) = I_{p_i,q_i:r}^{m,n}\left[\, x \,\middle|\, \begin{matrix} \{(a'_j, \alpha_j)_{1,n}\}, \{(a'_{ji}, \alpha_{ji})_{n+1,p_i})\} \\ \\ \{(b'_j, \beta_j)_{1,m}\}, \{(b'_{ji}, \beta_{ji})_{m+1,q_i}\} \end{matrix} \right]. \tag{1.5}$$

Defining

$$H_1(x) = \int_0^x H(x) \, dx, \tag{1.6}$$

and

$$K_1(x) = \int_0^x K(x) \, dx. \tag{1.7}$$

The reciprocity of $H(x)$ and $K(x)$ will be established by using mean square convergence method in the following theorem.

## 2. Result

**Theorem 2.1.** For $i = 1, 2, \ldots, r$,

(i) $q_i - 2m = p_i - 2n > 0$;

(ii) $\alpha_j \geq 0$ for $j = 1, 2, \ldots, n$; $\beta_j \geq 0$ for $j = 1, 2, \ldots, m$;
$\alpha_{ji} \geq 0$ for $j = n+1, n+2, \ldots, p_i$; $\beta_{ji} \geq 0$ for $j = m+1, m+2, \ldots, q_i$;

(iii) $$\frac{1}{2}D_i = \sum_{j=n+1}^{p_i} \alpha_{ji} - \sum_{j=1}^{n} \alpha_j + \sum_{j=m+1}^{q_i} \beta_{ji} - \sum_{j=1}^{m} \beta_j > 0. \tag{2.1}$$

Then the formulae

$$\frac{d}{dx} \int_0^\infty H_1(xu) f(u) \frac{du}{u} = g_h(x) \tag{2.2}$$

and

$$\frac{d}{dx} \int_0^\infty K_1(xu) f(u) \frac{du}{u} = g_k(x) \tag{2.3}$$

hold.

Defining almost everywhere functions $g_h(x)$ and $g_k(x)$ respectively, both belonging to $L^2(0, \infty)$. Also the reciprocal formulae

$$\frac{d}{dx} \int_0^\infty H_1(xu) g_h(u) \frac{du}{u} = f(x), \tag{2.4}$$

$$\frac{d}{dx} \int_0^\infty K_1(xu) g_k(u) \frac{du}{u} = f(x), \tag{2.5}$$

hold almost everywhere.

Further

$$\int_0^\infty |f(x)|^2 \, dx = \int_0^\infty g_h(x) g_k(x) \, dx. \tag{2.6}$$

**Proof.** If $h(s)$ and $k(s)$ are the Mellin transforms of $H(x)$ and $K(x)$, respectively, we get

$$h(s) = \mathcal{M}\{H(x)\} = \sum_{i=1}^{r} h_i(s) = \sum_{i=1}^{r} \left\{ \frac{\prod_{j=m+1}^{q_i} \Gamma(b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(1 - a_{ji} - \alpha_{ji} s)}{\prod_{j=1}^{m} \Gamma(1 - b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(a_j + \alpha_j s)\}} \right\} \tag{2.7}$$

and

$$k(s)=\mathcal{M}\{K(x)\}=\frac{\prod_{j=1}^{m}\Gamma(b'_j+\beta_j s)\ \prod_{j=1}^{n}\Gamma(1-a'_j-\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b'_{ji}-\beta_{ji}s)\ \prod_{j=n+1}^{p_i}\Gamma(a'_{ji}+\alpha_{ji}s)\right\}}. \tag{2.8}$$

It can be seen easily under the prescribed conditions, the functions $h(s)$ and $k(s)$ satisfy functional relation

$$h(s)\,k(1-s)\ =\ 1, \tag{2.9}$$

if

$$\begin{array}{lllll} a'_j &=& 1-a_j-\alpha_j & \text{and} & a'_{ji} = 1-a_{ji}-\alpha_{ji} \\ b'_j &=& 1-b_j-\beta_j & \text{and} & b'_{ji} = 1-b_{ji}-\beta_{ji} \end{array} \tag{2.10}$$

The satisfaction of the relation (2.9) alone is not sufficient to prove reciprocity of above kernels. So we have to verify that all the requirements of theorem given by Hardy and Titchmarsh [3; p.126] are fulfilled.

First of all we estimate the asymptotic behaviour of the $h(s)$ and $k(s)$ where $s=\sigma+it$, $\sigma$ and $t$ real and $|t|$ is large. For large $s$ the asymptotic behaviour of the Gamma function given by Whittaker and Watson [11; p.278] has been employed.

$$\log\Gamma(s+a)\ =\ \left(s+a-\frac{1}{2}\right)\log s-s+\frac{1}{2}\log(2\pi)+O(s^{-1}),\ \text{ where } |\arg s|<\pi. \tag{2.11}$$

The asymptotic behaviour of $h_i(s)$, the Mellin transform of $H$-function $H_i(x)$, is well established in the literature [1,7]. Thus we see that under the given conditions of the present theorem

$$h_i(s)x^{-s}\ =\ |t|^{D_i(\sigma-1/2)}e^{it(D_i\log|t|-\log x-B)}(Q+O|t|^{-1}) \tag{2.12}$$

for large $|t|$, where $B$ and $Q$ are constants.

This shows that integrand in (1.3) is bounded function of $t$ for large $t$ and consequently the integral is uniformly convergent.

Now to discuss the asymptotic behaviour of $k(s)$ making use of the theorem given by Saxena [8; p.58], we get

$$\begin{aligned} \phi(s)\ &=\ \left\{E_1+\frac{F_1}{s}+O\left(\frac{1}{|s|^2}\right)\right\}\Gamma(s)\cos\left(\frac{1}{2}\pi s\right),\quad t\to\infty \\ &=\ \left\{E_2+\frac{F_2}{s}+O\left(\frac{1}{|s|^2}\right)\right\}\Gamma(s)\cos\left(\frac{1}{2}\pi s\right),\quad t\to\infty, \end{aligned} \tag{2.13}$$

where $E_1, F_1, E_2$ and $F_2$ are constants depending $C_{1,r}, D_{1,r}, C_{2,r}$ and $D_{2,r}$; $(r = 1, 2, \ldots, n)$. Thus

$$L_i(s) = h_i(1-s)^{-1} = \left[ \frac{\prod_{j=m+1}^{q_i} \Gamma(b_{ji} + \beta_{ji} - \beta_{ji}s) \prod_{j=n+1}^{p_i} \Gamma(1 - a_{ji} - \alpha_{ji} + \alpha_{ji}s)}{\prod_{j=1}^{m} \Gamma(1 - b_j - \beta_j + \beta_j s) \prod_{j=1}^{n} \Gamma(a_j + \alpha_j - \alpha_j s)} \right]^{-1}. \tag{2.14}$$

We get

$$\phi(s) = \left[ \sum_{i=1}^{r} \frac{1}{L_i(s)} \right]^{-1} = \frac{\prod_{j=1}^{m} \Gamma(b'_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a'_j - \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b'_{ji} + \beta_{ji}s) \prod_{j=n+1}^{p_i} \Gamma(a'_{ji} + \alpha_{ji}s) \right\}} = k(s). \tag{2.15}$$

Therefore, transform $k(s)$ is a bounded function of $s$. Hence the integral in (1.5) is also uniformly convergent.

Using definition of $H_1(x)$ and $K_1(x)$ given in equations (1.6) and (1.7), we obtain

$$H_1(x) = \frac{1}{2\pi\omega} \int_L \frac{h(s)}{(1-s)} x^{1-s}\, ds, \tag{2.16}$$

and

$$K_1(x) = \frac{1}{2\pi\omega} \int_L \frac{k(s)}{(1-s)} x^{1-s}\, ds. \tag{2.17}$$

These have been proved to be valid only when $\sigma < 1/2$. As explained by Kesarwani [7], these formulae can be extended to the case $\sigma = 1/2$ by using the asymptotic behaviour of $h(s)$ and $k(s)$.

Now from (2.12) to (2.15) we see that $h(1/2+it)$ and $k(1/2+it)$ are bounded functions of $t$ satisfying the functional relation (2.9). The second condition requires that $H_1(x)$ and $K_1(x)$ be related with $h(s)$ and $k(s)$ as defined by (2.7) and (2.8). These obviously hold by (2.16) and (2.17). Finally $f(x) \in L^2(0, \infty)$ is covered by the hypothesis of the theorem.

Since all the conditions of the theorem [3; p.126] are satisfied, it follows that the formulae (2.2) to (2.6) are valid. Also it is obvious that (2.2) and (2.3) are equivalent, so that $H(x)$ and $K(x)$ are reciprocal kernels.

## 3. Special Cases

Putting $r = 1$, the kernels $K(x)$ and $H(x)$ reduce to the unsymmetrical Fourier kernels obtained by Kesarwani [7].

When all the parameters $\alpha$'s and $\beta$'s are equal to unity and $r = 1$, these kernels takes the form of unsymmetrical Fourier kernels involving $G$-functions [6].

## References

[1] Fox, C., *The G and H-functions as symmetrical Fourier kernels*, Trans. Amer. Math. Soc. **98** (1961), 53–62.

[2] Gupta, K.C., *On unsymmetrical Fourier kernels*, Vijnana Parishad Anusandhan Patrika **19** (1966 a), 207–211.

[3] Hardy, G.H. and Titchmarsh, E.C., *A class of Fourier kernels*, Proc. London Math. Soc. II **35** (1935), 116–155.

[4] Jain Deepak and Jain, Renu, *Some integrals involving Kampe de Feriet's series and I-function*, Kyungpook Math. J. **36** (1996), 53–62.

[5] Kesarwani, R.N., *G-functions as unsymmetrical Fourier kernels I*, Proc. Amer. Math. Soc. **13** (1962), 950–959.

[6] Kesarwani, R.N., *G-functions as unsymmetrical Fourier kernels II & III*, Proc. Amer. Math. Soc. **14** (1963), 18–28 and 271–277.

[7] Kesarwani, R.N., *A pair of unsymmetrical Fourier kernels*, Trans. Amer. Math. Soc. **115** (1965), 356–369.

[8] Saxena, V.P., *On a new class of kernels*, Jnanbha **1** (1971), 57–62.

[9] Saxena, V.P., *Formal solution of certain new pair of dual integral equations involving H-functions*, Proc. Nat. Acad. Sci. India **52**(A), III (1982), 366–375.

[10] Varma, R.C., Jain Renu and Vaishya, G.D., *Certain properties of the I-function*, Proc. Nat. Acad. Sci. India **59**(A),II (1989), 329–337.

[11] Whittaker, E.T. and Watson, G.N., *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1935.