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RADIAL VIBRATION OF MAGNETO-VISCO-ELASTIC CYLINDRICAL SHELL

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Abstract: This paper investigates the radial vibration of magneto visco-elastic shell. The material of the shell being aeolotropic and density ρ of the shell varies as $\rho = \rho_0 r^n$, where ρ_0 is constant and n is any integer. Lastly, frequency equation have been derived.

Keywords and Phrases: Radial vibration, magneto visco-elastic shell, frequency equation
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1. Introduction

Recently, a great deal of activity has emerged in the study of interaction of elastic and electromagnetic fields due to their extensive applications in science and technology. Kaliski [1], Narain [3,4], Narain and Srivastava [5], Narain and Verma [6,7], Nowacki [8], Nowacki and Kaliski [9], Paria [10] and many other have investigated the problems concerning elastic and electromagnetic fields. Sequal to there, the present paper in an attempt to investigate radial vibration of megneto-visco-elastic shell. The material of the shell being aeolotropic and density of the shell to be varying as the integral power of radius vector in the form $\rho = \rho_0 r^n$ where ρ_0 is constant and n is any integer. Frequency equation in several cases have been derived.

2. Fundamental Equations and Boundary Conditions

We consider aeolotropic visco-elastic prefectly conducting cylindrical shell of inner radius r_1 and outer radius r_2, and assumed that the space outside the shell to be surrounded by vacuum. We also consider that the boundary of the shell is mechanically stressed free. Initially, there exists an axial magnetic field of intensity $\vec{H}$ in the shell. The constituting relation for aeolotropic visco-elastic bodies in cylindrical co-ordinates (r, θ, z) may be written as

$$\begin{aligned}\sigma_{rr} &= \left(\lambda + \lambda' \frac{\partial}{\partial t} + \lambda'' \frac{\partial^2}{\partial t^2}\right) e_{rr} \\ \sigma_{\theta\theta} &= \left(\lambda + \lambda' \frac{\partial}{\partial t} + \lambda'' \frac{\partial^2}{\partial t^2}\right) e_{\theta\theta} \\ \sigma_{zz} &= \left(\lambda + \lambda' \frac{\partial}{\partial t} + \lambda'' \frac{\partial^2}{\partial t^2}\right) e_{zz}\end{aligned} \tag{2.1}$$

where σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and e_{rr}, $e_{\theta\theta}$, e_{zz} are the components of stress and strain respectively, λ, λ', λ'' are material constants. The equation of motion of magneto-elasticity for a perfect conductor with unit permeability as given by Kaliski [1] are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{4\pi}\Big\{rot.rot(\vec{u} \times \vec{H}\Big\} \times \vec{H} \;=\; \rho\,\frac{\partial^2 u_r}{\partial t^2} \tag{2.2}$$

$$\vec{E} \;=\; -\frac{1}{c}\frac{\partial \vec{u}}{\partial t} \times \vec{H}, \qquad \vec{h} \;=\; rot.(\vec{u} \times \vec{H}) \tag{2.3}$$

where $\vec{u}$ is the mechanical displacement vector. $\vec{E}$ is the electric intensity vector and $\vec{h}$ the perturbation in the magnetic intensity vector.

The electromagnetic field equations is vacuum are

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}^* \;=\; 0; \tag{2.4}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{h}^* \;=\; 0; \tag{2.5}$$

$$rot\,.\,\vec{E}^* \;=\; -\frac{1}{c}\frac{\partial \vec{h}^*}{\partial t}; \tag{2.6}$$

$$rot\,.\,\vec{h}^* \;=\; -\frac{1}{c}\frac{\partial \vec{E}^*}{\partial t}, \tag{2.7}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}$; $\vec{E}^*$, $\vec{h}^*$ denote the value of $\vec{E}$ and $\vec{h}$ respectively in vaccum. The components of strain as gives in Love [2] are

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad e_{zz} = \frac{\partial u_z}{\partial z}, \quad 2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}$$

$$2e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad 2e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

For radial vibration, we have

$$u_\theta \;=\; u_z \;=\; 0, \quad u_r \;=\; U(r)e^{iwt} \tag{2.8}$$

The components of strain take the form

$$e_{rr} = \frac{\partial u}{\partial r} e^{iwt}, \quad e_{\theta\theta} = \frac{u}{r} e^{iwt}, \quad e_{zz} = 0 \tag{2.9}$$

Also,

$$\begin{aligned} e_{\theta z} &= e_{rz} = e_{r\theta} = 0 \\ h_r^* &= h_\theta^* = 0, \quad h_z^* = h^* = V(r)e^{iwt} \\ H_r &= H_\theta = 0, \quad H_z = H \\ E_r^* &= E_\theta^* = 0, \quad E_z^* = We^{iwt} \end{aligned} \tag{2.10}$$

where V and W are functions of r alone.

The equation (2.3) gives

$$\begin{aligned} E &= -\frac{H_1}{c}\frac{\partial u}{\partial t} = -\frac{iw}{c} H_1 U e^{iwt} \\ h &= -\frac{H_1}{r}\frac{\partial (ru)}{\partial r} = -H_1\left(\frac{\partial U}{\partial r} + \frac{U}{r}\right) e^{iwt}. \end{aligned} \tag{2.11}$$

From (2.5), (2.7) and (2.10) we get

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{w^2}{c^2} U = 0 \tag{2.12}$$

$$W = \frac{ic}{w}\frac{\partial U}{\partial r}. \tag{2.13}$$

The boundary conditions are given by

$$\begin{aligned} \sigma_{rr} + T_{rr} &= T_{rr}^* \quad &\text{on} \quad r = r_1; \\ \sigma_{rr} + T_{rr} &= T_{rr}^* \quad &\text{on} \quad r = r_2; \\ \vec{E} &= \vec{E}^* \quad &\text{on} \quad r = r_1; \\ \vec{E} &= \vec{E}^* \quad &\text{on} \quad r = r_2, \end{aligned} \tag{2.14}$$

where T_{rr}, T_{rr}^* are Maxwellian tensors in the shell and vacuum respectively and may be expressed as

$$\begin{aligned} T_{rr} &= -\frac{H_1}{4\pi} h = \frac{H_1^2}{4\pi}\left(\frac{\partial U}{\partial r} + \frac{U}{r}\right) e^{iwt} \\ T_{rr}^* &= -\frac{H_1}{4\pi} h^* = \frac{H_1}{4\pi} U e^{iwt}. \end{aligned} \tag{2.15}$$

Therefore, the stress equation (2.1) with the help of equations (2.8) and (2.9) takes the form

$$\sigma_{rr} = \left(\lambda + \lambda' \frac{\partial}{\partial t} + \lambda'' \frac{\partial^2}{\partial t^2}\right) \frac{\partial U}{\partial r} e^{iwt} \tag{2.16}$$

$$\sigma_{\theta\theta} = \left(\lambda + \lambda' \frac{\partial}{\partial t} + \lambda'' \frac{\partial^2}{\partial t^2}\right) \frac{U}{r} e^{iwt}. \tag{2.17}$$

In case of radial vibration

$$\frac{1}{4\pi}\left\{rot.rot(\vec{u} \times \vec{H}\right\} \times \vec{H} = \frac{H^2}{4\pi}\left\{\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{U}{r^2}\right\} e^{iwt}. \tag{2.18}$$

3. Method of Solution

Suppose that the material density ρ varies as

$$\rho = \rho_0 r^n \tag{3.1}$$

where ρ_0 is constant and n is any integer, using equations (2.16), (2.17) and (2.18) the equation (2.2) gives

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{U}{r^2} + \frac{\rho_0 r^n w^2 U}{\left(K + \frac{H^2}{4\pi}\right)} = 0 \tag{3.2}$$

where

$$K = \lambda + iw\lambda' - w^2\lambda'' \tag{3.3}$$

using the transformation,

$$x = \frac{2}{n+2} r^{(n+2)/2} \tag{3.4}$$

the equation (3.2), takes the form

$$\frac{\partial^2 U}{\partial x^2} + \frac{1}{x}\frac{\partial U}{\partial x} + \left\{\mu^2 - \frac{\alpha^2}{x^2}\right\} U = 0 \tag{3.5}$$

where

$$\mu^2 = \frac{\rho_0 w^2}{\left(K + \frac{H^2}{4\pi}\right)}, \qquad \alpha^2 = \frac{2}{n+2}. \tag{3.6}$$

The solution of the equation (3.5) is given by

$$U = AJ_\alpha(\mu x) + BY_\alpha(\mu x) \tag{3.7}$$

where J_α and Y_α are the Bessel's functions of first and second kind of order α, A and B are constants.

Using equations (2.9) and (3.7), the equations (2.1) and (2.15) take the form

$$\sigma_{rr} = (\lambda + \lambda' iw - w^2 \lambda'') \left\{ A r^{n/2} J_{\alpha-1} \left(\frac{2\mu r^{(n+2)/2}}{n+2} \right) + B r^{n/2} Y_{\alpha-1} \left(\frac{2\mu r^{(n+2)/2}}{n+2} \right) \right\} e^{iwt} \tag{3.8}$$

and

$$T_{rr} = \frac{H^2}{4\pi} \left[\begin{array}{l} A \left\{ \mu r^{n/2} J_{\alpha-1} \left(\frac{2\mu}{n+2} r^{(n+2)/2} \right) + \frac{1}{r} J_\alpha \left(\frac{2\mu}{n+2} r^{(n+2)/2} \right) \right\} \\ +B \left\{ \mu r^{n/2} Y_{\alpha-1} \left(\frac{2\mu}{n+2} r^{(n+2)/2} \right) + \frac{1}{r} Y_\alpha \left(\frac{2\mu}{n+2} r^{(n+2)/2} \right) \right\} \end{array} \right] e^{iwt}. \tag{3.9}$$

Again, the solution of the equation (2.12) is given by

$$U = C J_0 \left(\frac{wr}{c} \right) + D Y_0 \left(\frac{wr}{c} \right), \tag{3.10}$$

where C and D are constants and J_0, Y_0 are Bessel's functions of order zero. Hence, the Maxwellian tensor T^*_{rr} in vacuum is given by

$$T^*_{rr} = \frac{H}{4\pi} U e^{iwt} = \frac{H}{4\pi} \left\{ C J_0 \left(\frac{wr}{c} \right) + D Y_0 \left(\frac{wr}{c} \right) \right\} e^{iwt} \tag{3.11}$$

thus,

$$T^*_{rr} = \left\{ \begin{array}{ll} \frac{H}{4\pi} C J_0 \left(\frac{wr}{c} \right) & \text{on } r \geq r_2 \\ \frac{H}{4\pi} D Y_0 \left(\frac{wr}{c} \right) & \text{on } r \leq r_2 \end{array} \right\} \tag{3.12}$$

Using equation (3.8), the equation (2.11) gives

$$E = -\frac{iw}{c} H U e^{iwt} = -\frac{iwH}{c} \left\{ A J_\alpha \left(\frac{2\mu}{n+2} r^{(n+2)/2} \right) + B Y_\alpha \left(\frac{2\mu}{n+2} r^{(n+2)/2} \right) \right\} e^{iwt}. \tag{3.13}$$

Using equation (3.7) in equation (2.13) we get

$$E^* = w e^{iwt} = \frac{ic}{w} \frac{\partial U}{\partial r} e^{iwt} = -i \left\{ C J_1 \left(\frac{wr}{c} \right) + D Y_1 \left(\frac{wr}{c} \right) \right\} e^{iwt}. \tag{3.14}$$

Hence,

$$E^* = \left\{ \begin{array}{ll} -iC J_1 \left(\frac{wr}{c} \right) & \text{on } r \geq r_2 \\ -iD Y_1 \left(\frac{wr}{c} \right) & \text{on } r \leq r_2 \end{array} \right\}. \tag{3.15}$$

Using equations (3.8), (3.9) and (3.2), the equation (2.14) gives

$$AX_{11} + BX_{12} + DX_{14} = 0 \tag{3.16}$$

$$AX_{21} + BX_{22} + CX_{23} = 0. \tag{3.17}$$

Using the equation (3.13) and (3.5), the equation (2.14) gives

$$AX_{31} + BX_{32} + DX_{34} = 0 \tag{3.18}$$

$$AX_{41} + BX_{42} + CX_{43} = 0. \tag{3.19}$$

where

$$
\begin{aligned}
X_{11} &= L_1 \left\{ \mu r_1^{n/2} J_{\alpha-1} \left(\frac{2\mu}{n+2}\, r_1^{(n+2)/2} \right) \right\}, \\
X_{12} &= L_1 \left\{ \mu r_1^{n/2} Y_{\alpha-1} \left(\frac{2\mu}{n+2}\, r_1^{(n+2)/2} \right) \right\}, \\
X_{14} &= \frac{H}{4\pi}\, Y_0 \left(\frac{\omega r_1}{c} \right), \\
X_{21} &= \mu r_2^{n/2} J_{\alpha-1} \left(\frac{2\mu}{n+2}\, r_2^{(n+2)/2} \right) + \frac{1}{r_2} J_\alpha \left(\frac{2\mu}{n+2}\, r_2^{(n+2)/2} \right), \\
X_{22} &= \mu r_2^{n/2} Y_{\alpha-1} \left(\frac{2\mu}{n+2}\, r_2^{(n+2)/2} \right) + \frac{1}{r_2} Y_\alpha \left(\frac{2\mu}{n+2}\, r_2^{(n+2)/2} \right), \\
X_{23} &= -\frac{H}{4\pi}\, J_0 \left(\frac{\omega r_1}{c} \right), \\
X_{31} &= -\frac{wH}{c}\, J_\alpha \left(\frac{2\mu}{n+2}\, r_2^{(n+2)/2} \right), \\
X_{32} &= -\frac{wH}{c}\, Y_\alpha \left(\frac{2\mu}{n+2}\, r_2^{(n+2)/2} \right), \\
X_{34} &= Y_1 \left(\frac{\omega r_1}{c} \right), \\
X_{41} &= \frac{wH}{c}\, J_\alpha \left(\frac{2\mu}{n+2}\, r_2^{(n+2)/2} \right), \\
X_{42} &= \frac{wH}{c}\, Y_\alpha \left(\frac{2\mu}{n+2}\, r_2^{(n+2)/2} \right), \\
X_{43} &= J_1 \left(\frac{\omega r_2}{c} \right).
\end{aligned}
$$

Eliminating A, B, C, D from equations (3.16),(3.17),(3.18) and (3.19) we get

$$
\begin{vmatrix}
X_{11} & X_{12} & 0 & X_{14} \\
X_{21} & X_{22} & X_{23} & 0 \\
X_{31} & X_{32} & 0 & X_{34} \\
X_{41} & X_{42} & X_{43} & 0
\end{vmatrix} = 0
$$

On solving the determinants, we get

$$\frac{X_{14}\,X_{32}+X_{34}\,X_{12}}{X_{23}\,X_{43}-X_{23}\,X_{42}} = \frac{X_{11}\,X_{34}+X_{14}\,X_{31}}{X_{43}\,X_{21}-X_{41}\,X_{42}}$$
$$\frac{P}{Q} = \frac{R}{S} \tag{3.20}$$

where

$$P := \frac{H^2}{4\pi}\,\frac{\omega}{c}\,Y_0\left(\frac{\omega r_1}{c}\right)\,Y_\alpha\left(\frac{2\mu}{n+2}\,r_1^{(n+2)/2}\right)+Y_1\left(\frac{\omega r_1}{c}\right)\,Y_{\alpha-1}\left(\frac{2\mu}{n+2}\,r_1^{(n+2)/2}\right)$$

$$Q := -\frac{wH^2}{4\pi c}\,J_0\left(\frac{\omega r_2}{c}\right)\,Y_\alpha\left(\frac{2\mu}{n+2}\,r_2^{(n+2)/2}\right)$$
$$+J_1\left(\frac{\omega r_2}{c}\right)\left\{\mu r_2^{n/2}Y_{\alpha-1}\left(\frac{2\mu}{n+2}\,r_2^{(n+2)/2}\right)+\frac{1}{r_2}Y_\alpha\left(\frac{2\mu}{n+2}\,r_2^{(n+2)/2}\right)\right\}$$

$$R := \frac{H^2}{4\pi}\,\frac{\omega}{c}\,Y_0\left(\frac{\omega r_1}{c}\right)\,J_\alpha\left(\frac{2\mu}{n+2}\,r_1^{(n+2)/2}\right)+Y_1\left(\frac{\omega r_1}{c}\right)\,J_{\alpha-1}\left(\frac{2\mu}{n+2}\,r_1^{(n+2)/2}\right)$$

$$S := -\frac{wH^2}{4\pi c}\,J_0\left(\frac{\omega r_2}{c}\right)\,J_\alpha\left(\frac{2\mu}{n+2}\,r_2^{(n+2)/2}\right)$$
$$+J_1\left(\frac{\omega r_2}{c}\right)\left\{\mu r_2^{n/2}J_{\alpha-1}\left(\frac{2\mu}{n+2}\,r_2^{(n+2)/2}\right)+\frac{1}{r_2}J_\alpha\left(\frac{2\mu}{n+2}\,r_2^{(n+2)/2}\right)\right\}.$$

If x is small, we use the following relations

$$\begin{aligned} Y_0(x) &= O(\log x) = K\log x \\ Y_n(x) &= O\left(\frac{1}{x^n}\right) = \frac{K}{x^n}, \quad \text{where } K \text{ is constant;} \\ J_n(x) &= \frac{x^n}{2^n\sqrt{n+1}}. \end{aligned} \tag{3.21}$$

After simplification equation (3.20) together with equation (3.21) we get

$$\frac{\rho_0 H^2}{2\pi c^2\left(K+\dfrac{H^2}{4\pi}\right)^{1/2}\alpha}\left(\frac{r_2}{r_1}\right)^{\sqrt{2(n+2)}}w^4+\left\{-\frac{H^4}{8\pi^2c^2\alpha}\left(\frac{r_2}{r_1}\right)^{\sqrt{2(n+2)}}+\frac{H^2}{4\pi c^2\alpha}\left(\frac{r_2}{r_1}\right)^{\sqrt{2(n+2)}}\right.$$
$$\left.+\frac{H^4}{8\pi^2c^2\alpha}+\frac{H^2}{4\pi c^2\alpha}\right\}w^3+\left\{\frac{\rho_0 H^2 r_1^{n/2}}{(n+2)\pi\alpha\left(K+\dfrac{H^2}{4\pi}\right)^{1/2}}\left(\frac{r_2}{r_1}\right)^{\sqrt{2(n+2)}}\right.$$

$$+\frac{2\rho_0 r_1^{n/2}}{(n+2)\Big(K+\dfrac{H^2}{4\pi}\Big)^{1/2}}\Big(\frac{r_2}{r_1}\Big)^{\sqrt{2(n+2)}}+\frac{H^2(n+2)}{4\pi c^2\rho_0 r_2^{n/2}}\Big(K+\frac{H^2}{4\pi}\Big)^{1/2}\Bigg\}w^2$$

$$+\left\{\Big(\frac{r_2}{r_1}\Big)^{\frac{n+2}{2}+1}-\Big[\Big(\frac{r_1}{r_2}\Big)^{n/2}\Big(\frac{r_2}{r_1}\Big)\Big]^{\sqrt{2(n+2)}}\right\}$$

$$+\frac{H^2(n+2)\Big(K+\dfrac{H^2}{4\pi}\Big)^{1/2}}{\partial\pi\rho_0 r_1^{\frac{n+2}{2}+1}}+\frac{(n+2)\Big(K+\dfrac{H^2}{4\pi}\Big)^2}{\partial\rho_0 r_1^{\frac{(n+2}{2}+1}} \;=\; 0. \qquad (3.22)$$

If there were no magnetic field, i.e., $H = 0$ and $\lambda' = \lambda'' = 0$ then the frequency equation at for $n = 0 \Rightarrow \alpha = 1$ for aeolotgropic cylindrical shell as

$$w^2 \;=\; \frac{\lambda^2}{r_2^2\rho_0^2}. \qquad (3.23)$$

The presence of r_2 in the dinominator of frequency equation (3.23) indicates that the frequency decreases when r_2 increases. It also indicates that the frequency does not depend upon the inner radius of the shell.

The following table indicates the frequency of different material:

S. No.	Material	ρ_0	λ	r_2	w	$\omega = w \times 10^{11}$
1.	Copper	8.9	8.5×10^{11}	1	1.06×10^{11}	1.06
				2	0.53×10^{11}	0.53
				3	0.35×10^{11}	0.35
				4	0.27×10^{11}	0.27
2.	Steel	7.8	11.2×10^{11}	1	1.43×10^{11}	1.43
				2	0.73×10^{11}	0.73
				3	0.47×10^{11}	0.47
				4	0.36×10^{11}	0.36
3.	Aluminium	2.7	5.6×10^{11}	1	2.07×10^{11}	2.07
				2	1.03×10^{11}	1.03
				3	0.69×10^{11}	0.69
				4	0.51×10^{11}	0.51
4.	Glass	2.5	2.8×10^{11}	1	1.12×10^{11}	1.12
				2	0.56×10^{11}	0.56
				3	0.28×10^{11}	0.28
				4	0.14×10^{11}	0.14

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