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ON LAMBERT SERIES AND CONTINUED FRACTIONS

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Abstract: In this paper, an attempt has been made to establish certain results involving Lambert series and continued fractions.

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1. Introduction

It is now customary to define the basic hypergeometric series by

$$ {}_2\Phi_1[a,b;\,c;\,q;\,z] \;=\; {}_2\Phi_1\left[\begin{matrix} a,b & ;q;\ z \\ c & \end{matrix}\right] \;=\; \sum_{n=0}^{\infty} \frac{[a;q]_n\,[b;q]_n\ z^n}{[c;q]_n\,[q;q]_n}, \tag{1.1}$$

where

$$[a;q] = \begin{cases} 1, & n=0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & n=1,2,3,\dots \end{cases}$$

is the q-shifted factorial and it is assummed that $c \neq q^{-m}$ for $m = 0,1,2,\cdots$. Also, $|q| < 1$ and $|z| < 1$ for the converegence of the series (1.1).

The generalized bilateral basic hypergeometric series is defined by

$$ {}_r\Psi_r\left[\begin{matrix} a_1,a_2,\cdots,a_r & ;q;\ z \\ b_1,b_2,\cdots,b_r & ; \end{matrix}\right] \;=\; \sum_{n=-\infty}^{\infty} \frac{[a_1,a_2,\dots,a_r;q]_n\,z^n}{[b_1,b_2,\dots,b_r;q]_n} \tag{1.2}$$

where $\left|\dfrac{b_1,b_2,\dots,b_r}{a_1,a_2,\dots,a_r}\right| < |z| < 1$ for the convergence of (1.2) and $[a_1,a_2,a_3,\dots a_r;q]_n = [a_1;q]_n[a_2;q]_n\dots[a_r;q]_n$.

Also,

$$[a;q]_{-n} \;=\; \frac{(-)^n\ q^{n(n+1)/2}}{a^n[q/a;q]_n} \ \text{ and } \ [a;\,q]_\infty \;=\; \prod_{r=0}^{\infty}(1-aq^r).$$

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Other notations appearing in this paper have their usual meanings. Bailey's sum for a well-poised ${}_3\Psi_3$ is:

$$ {}_3\Psi_3\left[\begin{matrix} b\ \ ,\ c\ \ ,\ d & ;q;\ q/bcd \\ q/b,\ q/c,\ q/d & \end{matrix}\right] = \sum_{n=-\infty}^{\infty} \frac{[b,c,d;q]_n (q/bcd)^n}{[q/b,q/c,q/d\ ;q]_n} $$

$$ = \frac{[q,q/bc,q/bd,q/cd;q]_\infty}{[q/b,q/c,q/d,q/bcd\ ;q]_\infty}. \tag{1.3} $$

[3; App (II)(II.31)]

Taking $c = 1/b$ in (1.3) we have

$$ \sum_{n=-\infty}^{\infty} \frac{[d;q]_n\,(q/d)^n}{[q/d\ ;q]_n(1-bq^n)(1-q^n/b)} = \frac{[q;q]_\infty^2[q/bd,bq/d;q]_\infty}{\left(1-\frac{1}{b}\right)[q/b,b,q/d,q/d\ ;q]_\infty}. \tag{1.4} $$

As $d \to \infty$, (1.4) yields:

$$ \sum_{n=-\infty}^{\infty} \frac{(-)^n\,q^{n(n+1)/2}}{(1-bq^n)(1-q^n/b)} = \frac{[q;q]_\infty^2}{\left(1-\frac{1}{b}\right)[b,q/b\ ;q]_\infty}. \tag{1.5} $$

Replacing q by q^k, and then setting $b = q^i$ and $d = q^i$ in (1.4) we have

$$ \sum_{n=-\infty}^{\infty} \frac{[q^j;q^k]_n q^{n(k-j)}}{[q^{k-j};q^k]_n(1-q^{kn+j})(1-q^{kn-i})} = \frac{[q^k;q^k]_\infty^2[q^{k-i-j},q^{k+i-j};q^k]_\infty}{(1-q^{-i})[q^i,q^{k-i},q^{k-j},q^{k-j};q^k]_\infty}, \tag{1.6} $$

where $i, j \not\equiv 0 \pmod{k}$.

For $j = i$, (1.6) yields

$$ \sum_{n=-\infty}^{\infty} \frac{[q^i;q^k]_n q^{n(k-i)}}{[q^{k-i};q^k]_n(1-q^{kn-i})(1-q^{kn-i})} = \frac{[q^k;q^k]_\infty^3[q^{k-2i};q^k]_\infty}{(1-q^{-i})[q^i,q^{k-i},q^{k-i},q^{k-i};q^k]_\infty}. \tag{1.7} $$

We shall make use of (1.5),(1.6),(1.7) and following known results in our analysis. Denis [2] has also established similar results involving Lambert series and continued fractions.

$$ \frac{[q;q^2]_\infty}{[q^2;q^4]_\infty} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q+q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^3+q^6}{1+\ldots}. \tag{1.8} $$

[1; (6.2.22) p.150]

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$$\frac{[q^3;q^4]_\infty}{[q;q^4]_\infty} = \frac{1}{1-}\,\frac{q}{1+q^2-}\,\frac{q^3}{1+q^4-}\,\frac{q^5}{1+q^6-\ldots}. \tag{1.9}$$

[1; (7.1.2) p.179]

$$\frac{[q^2;q^3]_\infty}{[q;q^3]_\infty} = \frac{1}{1-}\,\frac{q}{1+q-}\,\frac{q^3}{1+q^2-}\,\frac{q^5}{1+q^3}\,\frac{q^7}{1+q^4-\ldots}. \tag{1.10}$$

[1; (7.1.1) p.179]

$$\frac{[q,q^5;q^6]_\infty}{[q^3;q^6]^2_\infty} = \frac{1}{1+}\,\frac{q+q^2}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^3+q^6}{1+\ldots}. \tag{1.11}$$

[1; (6.2.37) p.154]

$$\frac{[q,q^7;q^8]_\infty}{[q^3,q^5;q^8]_\infty} = \frac{1}{1+}\,\frac{q+q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^3+q^6}{1+\ldots}. \tag{1.12}$$

[1; (6.2.38) p.154]

$$\frac{[q;q^5]_\infty\,[q^4;q^5]_\infty}{[q^2;q^5]_\infty\,[q^3;q^5]_\infty} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\ldots}. \tag{1.13}$$

[1; (1.1.1) and(1.1.2) p.9]

2. Main Results

(i) Replacing q by q^2 and then setting $b = q$ in (1.5) we get:

$$\sum_{n=-\infty}^{\infty}(-)^n\frac{q^{n(n+1)}}{(1-q^{2n+1})(1-q^{2n-1})} = \frac{[q^2;q^2]^2_\infty}{(1-q^{-1})[q;q^2]^2_\infty}. \tag{2.1}$$

Again, replacing q by q^4 and then taking $b = q^2$ in (1.5) we find:

$$\sum_{n=-\infty}^{\infty}(-)^n\frac{q^{2n(n+1)}}{(1-q^{4n+2})(1-q^{4n-2})} = \frac{[q^4;q^4]^2_\infty}{(1-q^{-2})[q^2;q^4]^2_\infty}. \tag{2.2}$$

Dividing (2.2) by (2.1) and using (1.8) we get:

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty}(-)^n\frac{q^{2n(n+1)}}{(1-q^{4n+2})(1-q^{4n-2})}}{\displaystyle\sum_{n=-\infty}^{\infty}(-)^n\frac{q^{n(n+1)}}{(1-q^{2n+1})(1-q^{2n-1})}} = \frac{q[-q^2;q^2]^2_\infty}{(1+q)}\left\{\frac{[q;q^2]^2_\infty}{[q^2;q^4]^2_\infty}\right\}$$

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$$= \frac{q[-q^2;q^2]^2_\infty}{(1+q)}\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q+q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^3+q^6}{1+\ldots}\right\}^2. \tag{2.3}$$

(ii) Again, taking $k=4$ and $i=1$ in (1.7) we find:

$$\sum_{n=-\infty}^{\infty}\frac{[q;q^4]_n\ q^{3n}}{[q^3;q^4]_n(1-q^{4n+1})(1-q^{4n-1})} = \frac{[q^4;q^4]^3_\infty[q^2;q^4]_\infty}{(1-q^{-1})[q,q^4]_\infty[q^3;q^4]^3_\infty}. \tag{2.4}$$

Again, taking $k=4$ and $i=3$ in (1.7) we find:

$$\sum_{n=-\infty}^{\infty}\frac{[q^3;q^4]_n\ q^{n}}{[q;q^4]_n(1-q^{4n+3})(1-q^{4n-3})} = \frac{[q^4;q^4]^3_\infty(1-q^{-2})[q^2;q^4]_\infty}{(1-q^{-3})[q^3,q^4]_\infty[q;q^4]^3_\infty}. \tag{2.5}$$

Dividing (2.5) by (2.4) and using (1.9) we get:

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty}\frac{[q^3;q^4]_n\ q^{n}}{[q;q^4]_n(1-q^{4n+3})(1-q^{4n-3})}}{\displaystyle\sum_{n=-\infty}^{\infty}\frac{[q;q^4]_n\ q^{3n}}{[q^3;q^4]_n(1-q^{4n+1})(1-q^{4n-1})}} = -\frac{(1-q)(1-q^2)[q^3;q^4]^2_\infty}{(1-q)^3\ [q;q^4]^2_\infty}$$

$$= -\frac{(1-q)(1-q^2)}{(1-q^3)}\left\{\frac{1}{1-}\,\frac{q}{1+q^2-}\,\frac{q^3}{1+q^4-}\,\frac{q^5}{1+q^6-\ldots}\right\}^2. \tag{2.6}$$

(iii) Taking $k=3$ and $i=2$ in (1.7) we get:

$$\sum_{n=-\infty}^{\infty}\frac{[q^2;q^3]_n\ q^{n}}{[q;q^3]_n(1-q^{3n+2})(1-q^{3n-2})} = \frac{q[q^3;q^3]^3_\infty}{(1+q)[q;q^3]^3_\infty}. \tag{2.7}$$

Again, taking $k=3$ and $i=1$ in (1.7) we get:

$$\sum_{n=-\infty}^{\infty}\frac{[q;q^3]_n\ q^{2n}}{[q^2;q^3]_n\ (1-q^{3n+1})(1-q^{3n-1})} = \frac{[q^3;q^3]^3_\infty}{(1-q^{-1})[q^2;q^3]^3_\infty}. \tag{2.8}$$

Dividing (2.7) by (2.8) and using (1.10) we get:

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty}\frac{[q^2;q^3]_n\ q^{n}}{[q;q^3]_n\ (1-q^{3n-2})(1-q^{4n-2})}}{\displaystyle\sum_{n=-\infty}^{\infty}\frac{[q;q^3]_n\ q^{2n}}{[q^2;q^3]_n(1-q^{3n+1})(1-q^{3n-1})}} = \frac{(1-q)^{-1}\,q[q^2;q^3]^3_\infty}{(1+q)\ [q;q^3]^3_\infty}$$

$$= -\frac{(1-q)}{(1+q)}\left\{\frac{1}{1-}\,\frac{q}{1+q-}\,\frac{q^3}{1+q^2-}\,\frac{q^5}{1+q^3-}\,\frac{q^7}{1+q^4-\ldots}.\right\}^2. \tag{2.9}$$

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(iv) Replacing q by q^6 and then setting $b = q^3$ in (1.5) we get:

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{3n(n+1)}}{(1-q^{6n+3})(1-q^{6n-3})} = \frac{[q^6;q^6]_\infty^2}{(1-q^{-3})[q^3\ ;q^6]_\infty^2}. \tag{2.10}$$

Again, replacing q by q^6 and then taking $b = q$ in (1.5) we have:

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{3n(n+1)}}{(1-q^{6n+1})(1-q^{6n-1})} = \frac{[q^6;q^6]_\infty^2}{(1-q^{-1})[q,q^5\ ;q^6]_\infty}. \tag{2.11}$$

Dividing (2.10) by (2.11) and using (1.11) we get:

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{3n(n+1)}}{(1-q^{6n+3})(1-q^{6n-3})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{3n(n+1)}}{(1-q^{6n+1})(1-q^{6n-1})}} = \frac{q^2}{(1+q+q^2)} \frac{[q,q^5;q^6]_\infty}{[q^3;q^6]_\infty^2}$$

$$= \frac{q^2}{(1+q+q^2)} \left\{ \frac{1}{1+} \frac{q+q^2}{1+} \frac{q^2+q^4}{1+} \frac{q^3+q^6}{1+\ldots} \right\}. \tag{2.12}$$

(v) Replacing q by q^8 and then setting $b = q$ in (1.5) we get:

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{4n(n+1)}}{(1-q^{8n+1})(1-q^{8n-1})} = \frac{[q^8;q^8]_\infty^2}{(1-q^{-1})[q,q^7\ ;q^8]_\infty}. \tag{2.13}$$

Again, replacing q by q^8 and then setting $b = q^3$ in (1.5) we have:

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{4n(n+1)}}{(1-q^{8n+3})(1-q^{8n-3})} = \frac{[q^8;q^8]_\infty^2}{(1-q^{-3})[q^3,q^5\ ;q^8]_\infty}. \tag{2.14}$$

Dividing (2.14) by (2.13) and using (1.12) we get:

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{4n(n+1)}}{(1-q^{8n+3})(1-q^{8n-3})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{4n(n+1)}}{(1-q^{8n+1})(1-q^{8n-1})}} = \frac{q^2}{(1+q+q^2)} \frac{[q,q^7;q^8]_\infty}{[q^3,q^5;q^8]_\infty}$$

$$= \frac{q^2}{(1+q+q^2)} \left\{ \frac{1}{1+} \frac{q+q^2}{1+} \frac{q^4}{1+} \frac{q^3+q^6}{1+\ldots} \right\}. \tag{2.15}$$

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(vi) Lastly, replacing q by q^5 and then taking $b = q$ in (1.5) we get:

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{5n(n+1)/2}}{(1-q^{5n+1})(1-q^{5n-1})} \;=\; \frac{[q^5;q^5]_\infty^2}{(1-q^{-1})[q,q^4\;;q^5]_\infty}. \tag{2.16}$$

Again, replacing q by q^5 and then setting $b = q^2$ in (1.5) we get:

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{5n(n+1)/2}}{(1-q^{5n+2})(1-q^{5n-2})} \;=\; \frac{[q^5;q^5]_\infty^2}{(1-q^{-2})[q^2,q^3\;;q^5]_\infty}. \tag{2.17}$$

Dividing (2.17) by (2.16) and using (1.13) we obtain:

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{5n(n+1)/2}}{(1-q^{5n+2})(1-q^{5n-2})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-)^n\, q^{5n(n+1)/2}}{(1-q^{5n+1})(1-q^{5n-1})}} \;=\; \frac{q}{1+q}\,\frac{[q,q^4;q^5]_\infty}{[q^2,q^3;q^5]_\infty}$$

$$= \left(\frac{q}{1+q}\right)\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1}\,\frac{q^4}{1+\ldots}\right\}. \tag{2.18}$$

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