South East Asian J. Math. & Math. Sc.
Vol.6 No.1(2007), pp.85–88

ON THE EXISTENCE OF PARTITIONS OF UNITY FOR PROXIMITY SPACES

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(*Received: February 02, 2006*)

Abstract: In the present paper it has been shown that if (X, δ) is a separated proximity space with a finite open covering u, then there exists a partition of unity dominated by u.

Keywords and Phrases: Proximity spacs, finite covering, partition of unity, Urysohn lemma, shrinking lemma
2000 AMS Subject Classification: 54A05, 54E05

1. Introduction and Preliminaries

As proximity is a structural layer distinct from and between topological structure and uniform structure, all proximity invariants are topological invariants, but some uniform invariants are not proximity invariants.

Proximity structure on a non-empty set X are defined in two ways. We can either specify when two sets are "close to one other" $(A\delta B)$ or when a set B is a proximal (also called uniform) neighbourhood of A $(B >> A)$. These two concepts are related by $A \not\delta B$ if and only if $A << X - B$.

In this paper preliminaries regarding proximity spaces and topological spaces can be found into [3,4,6]. However, few definitions and results that we need, are presented here for ready references.

Throughout the work, by a proximity space (X, δ) we shall mean a separated proximity space.

Let (X, δ) be a proximity space, For subsets A and B of X, we have

$$A << B \Rightarrow \; A << B^i, \quad (B^i - \text{interior of } B)$$

and $A << B \Rightarrow \bar{A} << B$, ($\bar{A}-$closure of A).

Lemma 1.1[5] (Proximal Urysohn lemma). Let (X, δ) be a proximity space. Let A, B be subsets of X such that $A \not\delta B$, then there exists a bounded proximally continuous map f such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Definition 1.1[6]. Let $\mathcal{R}$ and $\mathcal{T}$ be two families of subsets of X in a topological space (X, T). Then $\mathcal{T}$ is a said to be a refinement of $\mathcal{R}$ if and only if for every $S \in \mathcal{T}$ there exists an $R \in \mathcal{R}$ such that $S \subset R$.

The family $\mathcal{R}$ is called locally finite if and only if for every $x \in X$, there exists an N_x (neighbourhood of x) which has a nonempty intersection with at most a finite number of elements.

Note that a finite family is locally finite.

Definition 1.2[6]. A cover of a topological space X is a cover of the underlying point set of this set, i.e., any family u of subsets of X such that for each $x \in X$ there is an element $A \in u$ containing x. A cover u is called open if u consists of open sets.

Definition 1.3[3,6]. Let $\phi : X \to R$. Then the support of ϕ is defined to be the closure of the set $\{x \in X : \ \phi(x) \neq 0\}$.

Definition 1.4[6]. Let $u = \{U_1, U_2, \cdots U_n\}$ be a finite indexed open covering of the space X. An indexed family of continuous functions $\phi_i : X \to [0, 1]$ $i = 1, 2, \cdots, n$, is said to be partition of unity dominated by u if

(i) (Support ϕ_i) $\subset U_i$ for each i;

(ii) $\sum_{i=1}^{n} \phi_i(x) = 1$ for each x.

2. Existence of Partition of Unity

In this section we have first shown that in a separated proximity space for every finite open covering u of X there exist a finite open covering v of X such that $\bar{V}_i \subset U_i$ for each i, where $U_i \in u$ and $V_i \in v$. Using this result existence of partition of unity has been proved.

Definition 2.1[6]. A topological space (X, τ) is called bicompact if and only if every open cover of X has a finite subcover.

Thus a space is bicompact if and only if each open cover has a finite open refinement which is a cover of X.

Definition 2.2[3,6]. A topological space (X, τ) is called paracampact if every open cover of the space has a locally finite refinement which is an open cover of the space.

Every bicompact space is paracompact.

Lemma 2.1 (Shrinking lemma). Let (X, δ) be a proximity space where δ is given by $A\delta B$ if and only if $\bar{A} \cap \bar{B} \neq \phi$ and $u = \{U_1, U_2, \cdots U_n\}$ be an open covering of X. Then there exists an open cover $v = \{V_1, V_2, \cdots, V_n\}$ of X such that $V_i \subset U_i$ for each i.

Proof. We shall prove the lemma by induction. By hypothesis u covers X, therefore the set $A = X - (U_2 \cup U_3 \cup \cdots \cup U_n)$ is a closed subset of X which is contained in the $\tau(\delta)$ open set U_1. Now, $A \subset U_1$ implies $X - U_1 \subset X - A$, i.e., $(X - U_1) \cap A = \emptyset$. Hence $(X - U_1)\delta A$ i.e. $A << U_1$. Using strong axiom and using 1.2, we obtain a subset E_1 of X such that $A << E_1^i \subset \bar{E}_1 << U_1$, where E_1^i and $\bar{E}_1$ denote the interior and closure of E_1 respectively. Thus the collection $\{V_1, U_1, U_2, \cdots, U_n\}$ covers X, where $V_1 = E_1^i$.

Now for given subsets $E_2, E_3, \cdots, E_{k-1}$ of X such that the collection

$$\{E_1^i E_2^i, \cdots, E_{k-1}^i, U_k, U_{k+1}, \cdots, U_n\},$$

i.e., $\{V_1, V_2, \cdots, V_{k-1}, U_k, U_{k+1}, \cdots, U_n\}$, where $V_i = E_i^i$, $i = 1, \cdots, k-1$ covers X, let us write

$$B \ = \ X - (V_1 \cup V_2 \cup \cdots \cup V_{k-1}) - (U_{k+1} \cup U_{k+2} \cup \cdots \cup U_n),$$

then B is a closed subset of X contained in U_k. As above using strong axiom, we obtain a subset E_k of X such that

$$A << E_k^i \ \text{ and } \ \bar{E}_k << U_k$$

and the collection $\{E_1^i E_2^i, \cdots, E_k^i, U_{k+1}, \cdots, U_n\}$, i.e., $\{V_1, V_2, \cdots, V_k, U_{k+1}, \cdots, U_n\}$ covers X. At the n^{th} step the desired open cover $v = \{V_1, V_2, \cdots, V_n\}$ is obtained.

Theorem 2.1. Let $\{U_1, U_2, \cdots, U_n\}$ be a finite open covering of a separated proximity space (X, δ). Then there exists a partition of unity dominated by $\{U_i\}$.

Proof. Let $\{U_1, U_2, \cdots, U_n\}$ be a finite open covering of X. By the Lemma 2.1, there is an open covering say $\{V_1, V_2, \cdots, V_n\}$ of X such that

$$\bar{V}_i << U_i \ \text{ for each } i. \tag{2.1}$$

Again we can have open covering, say $\{W_1, W_2, \cdots, W_n\}$, of X such that

$$\bar{W}_i << V_i \ \text{ for each } i.$$

This implies $\bar{W}_i\delta(X - V_i)$. By Lemma 1.1, there exists a proximally continuous function $f_i : X \to [0, 1]$ such that $f_i(\bar{W}_i) = \{1\}$ and $f_i(X - V_i) = \{0\}$ for each i. Now since $X - V_i = f_i^{-1}\{0\}$, we get $f_i^{-1}(R - \{0\}) \subset V_i$ for each i. This gives $\overline{f_i^{-1}(R - \{0\})} \subset V_i$, i.e., (support f_i) $\subset \bar{V}_i \subset U_i$ by (2.1).

Obviously, $\psi(x) = \sum_{i=1}^{n} f_i(x) > 0$ for each x. Let us define $g_j(x) = \dfrac{f_j(x)}{\psi(x)}$.

Since each f_j is proximally continuous, the proximal continuity of ψ and hence of g_j for each j, follows by noting that finite sum and product of bounded proximally continuous functions are proximally continuous.

Thus the functions $\{g_1, g_2, \cdots, g_n\}$ from the desired partition of unity.

Remarks 2.1.

(i) The classical version of the above theorem for normal topological space (X, τ) follows by noting that the topology $\tau(\delta)$ induced by δ (as defined in Lemma 2.1) is such that $\tau = \tau(\delta)$.

(ii) It follows by the above theorem that every bicompact proximity space also possesses a partition of unity.

(iii) For paracompact proximity spaces the existence of partition of unity is proved by the method of transfinite construction and then using Zorn's Lemma.

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