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FIXED POINT THEOREMS FOR FOUR MAPPINGS IN METRIC SPACES

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Abstract: In this paper, we present a common fixed point theorem for weak compatible mapping of type (A) for four mappings in metric space which generalizes the result of Kang and Kim [6].

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1. Introduction

In Jungck [1] and in all generalization of Jungck's theorems, families of commuting mappings have been considered. Rhoades, Sessa and Khan [8] improved the results by assuming weak commutativity. Further, Jungck, Murthy and Cho [4] introduced the concept of compatible mappings of type (A) in metric space and improved the results of various authors. We use the idea of weak compatible mappings of type (A) in metric space as used by Pathak, Kang and Beak [7] in Menger and 2-metric spaces respectively which is equivalent to the concept of compatible and compatible mappings of type (A) under some conditions. In this section, we present a common fixed point theorem for weak compatible mapping of type (A) for four mappings in a metric space which generalizes the result of Kang and Kim [6].

2. Preliminaries

Definition 2.1. The pair (A, S) is said to be weak compatible of type (A) if

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \ \leq \ \lim_{n\to\infty} d(SAx_n, SSx_n)$$

and

$$\lim_{n\to\infty} d(SAx_n, AAx_n) \ \leq \ \lim_{n\to\infty} d(ASx_n, AAx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = t = \lim_{n\to\infty} Ax_n$ for some $t \in X$.

Preposition 2.1. Every pair of compatible mappings of type (A) is weak compatible of type (A).

Proof. Suppose that A and S are compatible mappings of type (A), therefore,

$$0 = \lim_{n\to\infty} d(ASx_n, SSx_n) \leq \lim_{n\to\infty} d(SAx_n, AAx_n)$$

and

$$0 = \lim_{n\to\infty} d(SAx_n, AAx_n) \leq \lim_{n\to\infty} d(ASx_n, AAx_n)$$

which shows that the pair $\{A, S\}$ is weak compatible of type (A).

Preposition 2.2. Let A and S are continuous mappings of a metric space (X, d) into self. If A and S are weak compatible of type (A), then they are compatible of type (A).

Proof. Suppose that A and S are weak compatible mapping of type (A). Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$.

Since A and S are continuous mappings, then we have

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \leq \lim_{n\to\infty} d(SAx_n, SSx_n) = d(St, St) = 0$$

and

$$\lim_{n\to\infty} d(SAx_n, AAx_n) \leq \lim_{n\to\infty} d(ASx_n, AAx_n) = d(At, At) = 0$$

Therefore A and S are compatible mappings of type (A) this completes the proof.

Preposition 2.3. Let A and S be weak compatible mappings of type (A) from metric space (X, d) into itself. If one of A and S is continuous, then A and S are compatible.

Proof. Without loss of generality, suppose that S is continuous. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \text{ for some } t \in X.$$

Since S is continuous, we have

$$\lim_{n\to\infty} SAx_n = St = \lim_{n\to\infty} SSx_n.$$

Now
$$\begin{aligned} d(SAx_n, ASx_n) &\le d(SAx_n, SSx_n) + d(SSx_n, ASx_n) \\ &\le 0 + d(SSx_n, ASx_n) \\ &= d(SSx_n, ASx_n) \end{aligned}$$

Since (A, S) are weak compatible of type (A). Therefore, we have

$$\begin{aligned} \lim_{n\to\infty} d(SAx_n, ASx_n) &\le \lim_{n\to\infty} d(ASx_n, SSx_n) \\ &\le \lim_{n\to\infty} d(SAx_n, SSx_n) \le 0. \end{aligned}$$

Therefore A and S are compatible.

Preposition 2.4. Let A and S be continuous mappings of (X, d) into itself. If A and S are compatible, then they are compatible of type (A).

As a direct consequence of Propositions 2.3 and 2.4, we have the following propositions.

Preposition 2.5. Let A and S be continuous mappings from a metric space (X, d) into itself. If A and S are compatible, then they are weak compatible of type (A).

Next we give some properties of weak compatible mappings of type (A).

Preposition 2.6. Let A and S be weak compatible maps of type (A) from metric space (X, d) into itself and let $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$, then we have the following:

(i) $\lim_{n\to\infty} ASx_n = St$ if S is continuous at t.

(ii) $\lim_{n\to\infty} SAx_n = At$ if A is continuous at t.

(iii) $SAt = ASt$ and $At = St$ if S and A are continuous at t.

Preposition 2.7. Let A and S be continuous mappings from a metric space (X, d) into itself. Then

(i) A and S are compatible of type A if and only if they are weak compatible of type (A).

(ii) A and S are compatible if and only if they are weak compatible of type (A).

Let A, B, S and T be self mappings from the metric space (X, d) into itself satisfying the following conditions:

$$A(X) \subseteq T(X), \quad B(X) \subset S(X) \tag{2.1}$$

$$d(Ax, By) \leq h \max\{d(Ax, Sx),\, d(By, Ty),\, \frac{1}{2}[d(Ax, Ty) + d(By, Sx)], d(Sx, Ty)\} \quad (2.2)$$

for all $x, y \in X$, where $0 \leq h < 1$.

Then for any arbitrary point x_0 in X by (2.1), we choose a point $x_1 \in X$ such that $Tx_1 = Ax_0$ and for this point x_1, we can choose a point $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on inductively we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1}. \quad (2.3)$$

Lemma 2.1[6]. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (2.1) and (2.2). Then the sequence $\{y_n\}$ defined by (2.3) is a Cauchy sequence in X.

Lemma 2.2. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (2.1),(2.2) and (2.3) $A(X) \cap B(X)$ is a complete subspace of X. Then pairs (A, S) and (B, T) have a coincidence point in X.

Proof. By Lemma 2.1, the sequence $\{y_n\}$ defined by (2.3) is a Cauchy sequence in $A(X) \cap B(X)$. Since $A(X) \cap B(X)$ is a complete subspace of X, so $\{y_n\}$ converges to a point w (say), in $A(X) \cap B(X)$.

On the other hand, since the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ are also Cauchy sequence in $A(X) \cap B(X)$, so they also converges to the same limit w. Hence there exist two points $u, v \in X$ such that $Au = w$ and $Bv = w$, respectively. By (2.2), we have

$$\begin{aligned} d(Su, y_{2n+1}) = &\ d(Su, Ax_{2n}) \\ \leq &\ h \max\{d(Su, Tu),\, d(Ax_{2n}, Bx_{2n+1}), \\ &\ \frac{1}{2}[d(Su, Bx_{2n+1}) + d(Tx_{2n+1}, Au)], d(Au, Tx_{2n+1})\}. \end{aligned}$$

Letting limit as $n \to \infty$, we have

$$\begin{aligned} d(Su, w) \leq &\ h \max\{d(Su, w),\, d(w, w), \frac{1}{2}[d(Su, w) + d(w, Au)], d(Au, w)\} \\ = &\ hd(Su, w), \end{aligned}$$

a contradiction. Hence $Au = w = Su$.

Similarly, we can show that v is also a coincidence point of B and T.

Lemma 2.3. Let A and S be weak compatible mappings of type (A) from a metric space (X, d) into itself. If $Au = Su$ for some $u \in X$, then $SAu = SSu = AAu = ASu$.

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n = u$, $n = 1, 2, 3, \ldots$ and $Su = Au$. Now, we have

$$\lim_{n\to\infty} Sx_n \;=\; Su \;=\; \lim_{n\to\infty} Ax_n.$$

Since S and A are weak compatible mappings of type (A), we have

$$\begin{aligned} d(SAu, AAu) &= \lim_{n\to\infty} d(SAx_n, AAx_n) \\ &\le \lim_{n\to\infty} d(ASx_n, AAx_n) \;=\; 0. \end{aligned}$$

Hence $SAu = AAu$. Therefore, $SAu = SSu = AAu = ASu$.

Kang and Kim [6] proved the following:

Theorem 2.1. Let A, B, S and T be self mappings from a metric space (X, d) into itself satisfying the conditions (2.1),(2.2) and (2.3). Suppose that

$$\text{One of } A,\ B,\ S \text{ and } T \text{ is continuous;} \tag{2.4}$$

$$\text{Pairs } [A, S] \text{ and } [B, T] \text{ are compatible on } X. \tag{2.5}$$

Then A, B, S and T have a unique common fixed point in X.

Now, we prove the following theorem:

Theorem 2.2. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (2.1),(2.2) and (2.3) and

$$(B, T) \text{ are weak compatible of type } (A). \tag{2.6}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. By Lemma 2.2, there exists two points u, v in X such that $Su = Au = w$ and $Tv = Bv = w$, respectively. Since A and S are weak compatible of type (A), then by Lemma 2.3, $SAu = SSu = AAu = ASu$, which implies that $Sw = Aw$. Similarly, we have $Tw = Bw$.

Now, we prove that $Sw = w$. Suppose $Sw \ne w$, then by (2.2), we have

$$\begin{aligned} d(Sw, y_{2n+1}) &= d(Sw, Tx_{2n+1}) \\ &\le h \max\{d(Sw, Aw),\, d(Tx_{2n+1}, Bx_{2n+1}), \\ &\quad \frac{1}{2}[d(Sw, Bx_{2n+1}) + d(Tx_{2n+1}, Aw)], d(Aw, Tx_{2n+1})\}. \end{aligned}$$

Proceeding limit as $n \to \infty$, we have

$$\begin{aligned} d(Sw, w) \leq\ & h \max\{d(Sw, w),\ d(w, w),\ \frac{1}{2}[d(Sw, w) + d(w, Aw)], d(Aw, w)\} \\ =\ & hd(Sw, w), \end{aligned}$$

a contradiction. Hence $Sw = w = Aw$.

Similarly, we have $Tw = w = Bw$. This means that w is a common fixed point of A, B, S and T.

Uniqueness. Let $z \neq w'$ be another common fixed point of A, B, S and T. Then we have

$$\begin{aligned} d(w', z) =\ & d(Sw', Tz) \\ \leq\ & h \max\{d(Sw', Aw'),\ d(Tz, Bz, \frac{1}{2}[d(Sw', Bz) \\ & + d(Tz, Aw')], d(Aw', Tz)\} \\ =\ & hd(Sw', Tz) \\ =\ & hd(w', z), \end{aligned}$$

a contradiction. Hence $w' = z$.

In support of our theorem, we give the following example.

Example 2.1. Let $X = [0, 1]$ with the Euclidean metric d. Define A, B, S and $T : X \to X$ by

$$Ax = x^3, \quad Bx = x^2, \quad Sx = 2x^6 - 1 \text{ and } Tx = 2x^4 - 1$$

for all x in X. Now $A(X) = B(X) = S(X) = T(X) = X$. Moreover, since

$$\begin{aligned} d(Ax_n, Sx_n) &= |2x_n^3 + 1|\,|x_n - 1| \to 0 \text{ iff } x_n \to 1, \\ \lim_{n\to\infty} d(ASx_n, SAx_n) &= \lim_{n\to\infty} 6x_n^6(x_n^6 - 1)^2 = 0 \text{ as } x_n \to 1. \end{aligned}$$

Also, since

$$\begin{aligned} d(Bx_n, Tx_n) &= (2x_n^2 + 1)|x_n^2 - 1| \to 0 \text{ iff } x_n \to 1, \\ \lim_{n\to\infty} (BTx_n, TBx_n) &= \lim_{n\to\infty} 2(x_2^4 - 1)^2 = 0 \text{ as } x_n \to 1. \end{aligned}$$

Clearly, (A, S) and (B, T) are weak compatible mappings of type (A). Further, we obtain

$$\begin{aligned} d(Ax, By) \leq\ & \frac{1}{4} d(Sx, Ty) \\ \leq\ & \frac{1}{4} \max\{d(Ax, Sx),\ d(By, Ty)\frac{1}{2}[d(Ax, Ty) \\ & + d(By, Sx)], d(Sx, Ty)\}, \end{aligned}$$

since
$$\begin{aligned} d(Sx,Ty) &= 2|x^3-y^2|\,|x^3+y^2| \\ &\geq 4d(Ax,By) \text{ for all } x,y. \end{aligned}$$

Therefore all the conditions of Theorem 2.2 are satisfied and 0 is the unique common point fixed point.

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