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ON DETERMINATION OF THE MINIMUM GENUS OF COMPACT RIEMANN SURFACES ADMITTING A CLASS OF METACYCLIC AUTOMORPHISM GROUPS

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Abstract: Every finite group can be represented as a group of automorphisms of a compact Riemann surface of genus $g \geq 2$. It is of interest to determine the minimum genus of the Riemann surface on which a given finite group acts as a group of automorphisms.

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1. Introduction

In this paper we find the minimum genus of the surface of which the group $G = D_m \times C_p$, where m is an odd integer greater than one and p is any odd prime, is a group of automorphisms. The minimum value of the genus g of a compact Riemann surface having G, as it's group of automorphisms is obtained as

(i) $g = 1 - m\dfrac{p+1}{2} + m\left(p - \dfrac{p}{pq_1}\right)$, if $(p,m) = 1$;

(ii) $g = 1 - m\dfrac{p+1}{2} + m\left(p - \dfrac{p}{q_1}\right)$, if $(p,m) \neq 1$;

(iii) $g = 4$, if $p = m = 3$;

where m has the prime decomposition

$$m = q_1^{r_1} q_2^{r_2} q_3^{r_3} \cdots q_l^{r_l}; \quad r_i > 0, \quad q_1 < q_2 < \cdots < q_l, \; q_i \geq 3.$$

Groups of automorphisms of compact Riemann surfaces constituted a glamorous topic of research during the last decade of the 20th century and for its beauty and elegance of the materials this topic continues to draw attention of the mathematicians even after a century. The automorphisms of compact Riemann surfaces of genus $g \geq 2$ form a finite group and every finite group can be represented as the group of automorphisms of some Riemann surfaces of genus $g \geq 2$ [1].

In the study of Riemann surface automorphisms, Fuchsian groups have an important role. A Fuchsian group is a discrete subgroup of the group of linear substitutions of the form $w = \dfrac{az+b}{cz+d},\ a, b, c, d \in R,\ ad - bc = 1$, which preserves the interior or exterior of a circle.

The study of groups of automorphisms of compact Riemann surfaces was simplified due to Riemann Uniformization theorem [1], which states that each Riemann surface of genus $g \geq 2$ can be represented as the orbit space D/Γ of the upper half complex plane $D = \{z \in C : \ \mathrm{Im}\, z \geq 0\}$ where Γ is a Fuchsian group.

A Fuchsian group Γ with compact orbit space has the following presentation:

Generator : $\quad x_1, x_2, x_3, \cdots, x_r,\ a_1, b_1, a_2, b_2, \cdots, a_r, b_r.$

$$\text{Relation : } \quad x_1^{m_1} = x_2^{m_2} = \cdots = x_r^{m_r} = \prod_{i=1}^{r} x_i \prod_{j=1}^{\gamma} a_j b_j a_j^{-1} b_j^{-1}. \qquad (1.1)$$

A Fuchsian group with presentation (1.1) is said to have signature

$$\Delta(\gamma; m_1, m_2, \cdots, m_r), \qquad (1.2)$$

with

$$\delta(\Gamma) = 2\gamma - 2 + \sum_{i=1}^{r} \left(l - \frac{1}{m_i}\right) > 0. \qquad (1.3)$$

The integers $m_i (\geq 2)$ are called-the periods and γ the genus of Γ.

A Fuchsian group without elements of finite order except the identity is called a surface group and if g is the genus of a Fuchsian surface group K, then

$$\delta(K) = 2(g-1). \qquad (1.4)$$

In view of (1.3), when $\gamma = 0$, the following signatures of Γ are untenable.

(i) (m, n) (ii) $(2, 2, m)$ (iii) $(2, 3, 3)$ (iv) $(2, 3, 4)$ (v) $(2, 3, 5)$
(vi) $(2, 3, 6)$ (vii) $(2, 2, 2, 2)$ (viii) $(3, 3, 3)$ (ix) $(2, 4, 4)$.

Some properties of Fuchsian groups are stated below:

I. If Γ_1 is a subgroup of Γ of finite index then it is known that Γ_1 is also a Fuchsian group and

$$[\Gamma : \Gamma_1] \;=\; \frac{\delta(\Gamma_1)}{\delta(\Gamma)}. \tag{1.5}$$

II. Any fmite order element of a Fuchsian group Γ is a conjugate of a finite order generator of Γ.

By Riemann uniformization theorem one can prove that a finite group G is the group of automorphisms of a compact Riemann surface of genus $g \geq 2$ if and only if there exists a Fuchsian group Γ with compact orbit space and an epimorphism $\phi : \Gamma \to G$ such that $\ker \phi$ is a surface group of genus g. Such an epimorphism is called a smooth epimorphism and the factor group is called a smooth quotient. If Γ has a presentation (1.1), we have from (1.5)

$$|G| \;=\; \left|\frac{\Gamma}{\ker \phi}\right| \;=\; \frac{\delta(\ker \phi)}{\delta(\Gamma)}$$

or

$$2(g-1) \;=\; |G|\left\{2\gamma - 2 + \sum\left(1 - \frac{1}{m_i}\right)\right\}. \tag{1.6}$$

It is of interest to determine the minimum genus of a surface on which a given finite group G acts as a group of automorphisms.

Minimization of the right hand expression of (1.6) for a given finite group G also minimizes the left hand expression giving the minimum genus g of a surface, which admits G as its group of automorphisms.

The minimum genus of a compact Riemann surface, which admits a cyclic group as its group of automorphisms was determined by Harvey [10]. The same problem for the class of non cyclic Abelian groups, zs-metacyclic groups, K-metacyclic groups, dihedral groups, PSL $(2, p)$ etc. were solved respectively by Machlaclan [14], Chetiya, Dutta and Patra [5], Chetiya and Patra [6], Chutiya [7] and Glover and Sjerve [9] respectively. Recently this minimum genus problem for a special class of metacyclic groups is solved by Michael [16].

Here in this paper we determine the minimum genus of the surface, which admits the group $G = D_m \times C_p$, m being an odd integer > 1 and p is any odd prime, as its group of automorphisms.

The result of Harvey obtained for cyclic groups and the result of Chutiya obtained for Dihedral groups can be deduced as corollaries from our result.

2. Preliminaries

The group $G = D_m \times C_p$ under consideration is a group generated by s and t satisfying

$$s^m = t^{2p} = (st)^{2p} = 1, \quad t^{-1}st = s^{-1} \quad [8] \tag{2.1}$$

where m and p are as stated in Section 1.

The elements of G are of the form $t^i s^j$, $1 \leq i \leq 2p$, $1 \leq j \leq m$ and the k-th power of any element $t^i s^j$ of G is

$$\begin{aligned}(t^i s^j)^k &= t^{ki} s^{j[(-1)^{(k-1)i}+(-1)^{(k-2)i}+\cdots+(-1)^i+1]} \\ &= t^{ki} s^{kj} \text{ if } i \text{ is even} \\ &= t^{ki} \text{ if } i \text{ is odd, } k \text{ is even} \\ &= t^{ki} s^j \text{ if both } i \text{ and } k \text{ are odd; } 1 \leq j \leq m\end{aligned}$$

Hence the orders of the elements of G are as follows:

(i) order of $t^i s^j = 2$ for $i = p$, $1 \leq j \leq m$.
$= 2p$ for any odd value of $i(\neq p)$, $1 \leq j \leq m$.

(ii) order of $t^i s^j = p_j$, where $p_j = [p, m_j]$, $m_j = \dfrac{m}{d_j}$, $d_j = (m, j)$, $1 \leq j \leq m$ when i is any even $< 2p$.

(iii) order of $s^j = m_j = \dfrac{m}{d_j}$, $d_j = (m, j)$, $1 \leq j \leq m - 1$,

where (a, b) and $[a, b]$ denote respectively the h.c.f and l.c.m. of a and b.

Note that when $(p, m) = 1$, then $p_j = pm_j$.

It can easily be varified that the commutator subgroup G' of G, is a cyclic group generated by an even power of s. Again the quotient group $G/G' \cong Z_{2p}$. Hence G is a z-metacyclic group [8].

We now find a set of necessary and sufficient conditions on the periods and genus of Γ for the existence of a smooth epimorphism $\phi : \Gamma \to G$.

3. Existence of Smooth Epimorphism

Let Γ be a Fuchsian group with presentation (1.1) and G be the group with presentation (2.1) and $(p, m) = 1$. Let $\phi : \Gamma \to G$ be a smooth epimorphism. We note that an epimorphism $\phi : \Gamma \to G$ will be smooth if and only if ϕ preserves the periods of Γ [1]. With this we get the following conditions for the existence of a smooth epimorphism $\phi : \Gamma \to G$.

1. If Γ is a surface group, then $\gamma \geq 2$ by (1.3).
2. The elements of G are of orders 2, $2p$, p, m_j, p_j as stated in Section 2. Since $\phi : \Gamma \to G$ preserves the periods of Γ, so the possible periods of Γ will take the values from $\{2, 2p, p, m_j, p_j\}$, $1 \leq j \leq m-1$ where m_j and p_j are defined as in Section 2.

Suppose Γ has the signature

$$\Gamma\Big(\gamma,\ \underbrace{2,2,\cdots,2}_{t_2 \text{ times}},\ \underbrace{2p,2p,\cdots,2p}_{t_{2p} \text{ times}},\ \underbrace{p,p,\cdots,p}_{t_p \text{ times}},\ \underbrace{m_j,m_j,\cdots,m_j}_{t_{m_j} \text{ times}},\ \underbrace{p_j,p_j,\cdots,p_j}_{t_{p_j} \text{ times}}\Big)$$

where $t_2,\ t_{2p},\ t_p, t_{mj},\ t_{pj}$ denote the number of occurrences of periods $2,\ 2p,\ p,\ m_j,\ p_j$ respectively. By (1.6) we get

$$g = 1 + pm\left[2(\gamma-1) + t_2(1-\frac{1}{2}) + t_{2p}(1-\frac{1}{2p}) + t_p(1-\frac{1}{p}) + \sum t_{m_j}(1-\frac{1}{m_j}) + \sum t_{p_j}(1-\frac{1}{p_j})\right]$$

$$= 1+2pm(\gamma-1)+m\left[\frac{1}{2}\Big\{pt_2+(2p-1)t_{2p}\Big\}\right]+m(p-1)t_p+p\left\{\sum\frac{m}{m_j}(m_j-1)t_{m_j} + \sum\frac{m}{p_j}(p_j-1)t_{p_j}\right\} \tag{3.1}$$

g being an integer, the expression on the right hand side of (3.1) will be an integer if t_2 and t_{2p} are of same parity and if either of them is zero, then the other must be even.

Now let x_i, y_i, z_i, u_i and v_i be the finite order generators of Γ of order $2,\ 2p,\ p,\ m_j,\ p_j$ respectively. As ϕ preserves the periods of Γ we must have from Section 2 the followings

$$\left.\begin{array}{lcll} \phi(x_i) &=& t^p s^j, & 1 \leq j \leq m \\ \phi(y_i) &=& t^a s^f, & 1 \leq f \leq m,\ \ a(\neq\ p) \text{ is an odd positive integer} < 2p \\ \phi(z_i) &=& t^{2k}, & k \text{ is an positive integer} < p \\ \phi(u_i) &=& s^q, & 1 \leq q \leq m-1 \\ \phi(v_i) &=& t^b s^h, & 1 \leq h \leq m-1,\ \ b \text{ is an even positive integer} < 2p. \end{array}\right\} \tag{3.2}$$

If $\alpha_i,\ \beta_i,\ i = 1, 2, \cdots \gamma$ are the infinite order generators of Γ, then from (1.1) we have

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_{2p}}\phi(y_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{m_j}}\phi(u_i)\prod_{i=1}^{t_{p_j}}\phi(v_i)\prod_{i=1}^{\gamma}\phi[\alpha_i,\beta_i] = 1. \tag{3.3}$$

By Section 2, $\phi[\alpha_i, \beta_i] = s^\delta$, where δ is an even integer, $1 \leq \delta \leq m$ and $[\alpha_i, \beta_i]$ denotes the commutator of α_i, β_i. If none of the t_i's is zero, then (3.2) and (3.3) gives

$$\prod_{i=1}^{t_2} t^p s^{j_i} \prod_{i=1}^{t_{2p}} t^{a_i} s^{f_i} \prod_{i=1}^{t_p} t^{2k} \prod_{i=1}^{t_{m_j}} s^{q_i} \prod_{i=1}^{t_{p_j}} t^{b_i} s^{h_i} \prod_{i=1}^{\gamma} s^\delta = 1. \tag{3.4}$$

where $1 \leq q_i,\ h_i \leq m-1$, $1 \leq j_i, f_i,\ \ \delta \leq m$.

When both t_2, t_{2p} are even, then

$$\left.\begin{array}{l} \sum_{i=1}^{t_{2p}} a_i + \sum_{i=1}^{t_p} 2k_i + \sum_{i=1}^{t_{p_j}} b_i \equiv 0 (\operatorname{mod} 2p) \\ \sum_{i=1}^{t_2} (-1)^i j_i + \sum_{i=1}^{t_{2p}} (-1)^{t_2+i} f_i + \sum_{i=1}^{t_{m_j}} q_i + \sum_{i=1}^{t_{p_j}} h_i + \sum_{i=1}^{\gamma} \delta_i \equiv 0 (\operatorname{mod} m) \end{array}\right\} \tag{3.4.1}$$

and when t_2, t_{2p} are odd, then

$$\left.\begin{array}{l} p + \sum_{i=1}^{t_{2p}} a_i + \sum_{i=1}^{t_p} 2k_i + \sum_{i=1}^{t_{p_j}} b_i \equiv 0 (\operatorname{mod} 2p) \\ \sum_{i=1}^{t_2} (-1)^i j_i + \sum_{i=1}^{t_{2p}} (-1)^{t_2+i} f_i + \sum_{i=1}^{t_{m_j}} q_i + \sum_{i=1}^{t_{p_j}} h_i + \sum_{i=1}^{\gamma} \delta_i \equiv 0 (\operatorname{mod} m) \end{array}\right\} \tag{3.4.2}$$

If a_i, k_i, b_i, j_i etc. are suitably choosen, then the above two congruences have solutions and thus $\phi : \Gamma \to G$ is a smooth homomorphism. From (3.2) it is clear that ϕ is also an epimorphism with $\gamma \geq 0$.

We now examine the cases when one or more of the t_i's are zero.

Case I. When one of the t_i's is zero, $\gamma \geq 0$ and $t_2 + t_{2p}$ is even, then clearly (3.2) and (3.3) define a smooth epimorphism $\phi : \Gamma \to G$.

Case II. When any two of the t_i's are zero. The possible cases are:

(a) $t_2 = t_p = 0$. Obviously t_{2p} is even.

From (3.4) we get

$$\begin{array}{l} \sum_{i=1}^{t_{2p}} a_i + \sum_{i=1}^{t_{p_j}} b_i \equiv 0 (\operatorname{mod} 2p) \\ \sum_{i=1}^{t_{2p}} (-1)^i f_i + \sum_{i=1}^{t_{m_j}} q_i + \sum_{i=1}^{t_{p_j}} h_i + \sum_{i=1}^{\gamma} \delta_i \equiv 0 (\operatorname{mod} m) \end{array}$$

We can choose a_i, k_i, b_i, j_i etc. such that the above congruences have a solution. So $\phi : \Gamma \to G$ will be a smooth epimorphism.

Similarly $\phi : \Gamma \to G$ defines a smooth epimorphism when $t_2 + t_{2p}$ is even and $\gamma \geq 0$ in the following cases

(b) $t_2 = t_{m_j} = 0$, (c) $t_2 = t_{p_j} = 0$, (d) $t_{2p} = t_{m_j} = 0$ and (e) $t_p = t_{m_j} = 0$.

Now we consider the cases

(f) $t_2 = t_{2p} = 0$, (g) $t_{2p} = t_p = 0$, t_2 is even,
(h) $t_{2p} = t_{p_j} = 0$, t_2 is even, (i) $t_p = t_{p_j} = 0$, $t_2 + t_{2p}$ is even .

In case (f), (3.2) defines a smooth homomorphism. But $\phi : \Gamma \to G$ will be a smooth epimorphim only if $\gamma \geq 1$. Because when $t_2 = t_{2p} = 0$, we see from (3.2), that no finite order element of Γ is mapped to $t \in G$ and as a result $\phi(\Gamma) \subset G$. So ϕ is not onto.

If $\gamma \geq 1$ we can exhibit a smooth epimorphism by letting $\phi(\alpha_1) = \phi(\beta_1) = t$ in (3.2).

In the case (g), (3.4) gives

$$\sum_{i=1}^{t_{p_j}} b_i \equiv 0 (\operatorname{mod} 2p) \quad \text{and} \quad \sum_{i=1}^{t_2} (-1)^i j_i + \sum_{i=1}^{t_{m_j}} q_i + \sum_{i=1}^{t_{p_j}} h_i + \sum_{i=1}^{\gamma} \delta_i \equiv 0 (\operatorname{mod} m)$$

Obviously the second congruence has solutions, but the first one has a solution only if $t_{p_j} \geq 2$ as $b_i < 2p$. So $\phi : \Gamma \to G$ is a homomorphism if $t_{p_j} \geq 2$ and from (3.2) it is clear that ϕ will be a smooth epimorphism for $\gamma \geq 0$.

Similarly in case (h), $\phi : \Gamma \to G$ will be a smooth epimorphism, if $t_p \geq 2$, $\gamma \geq 0$ and in case (i), $t_{2p} \geq 2$, $\gamma \geq 0$. Next we consider the case

(j) $t_{m_j} = t_{p_j} = 02$. Here also $t_2 + t_{2p}$ is even.

When t_2, t_{2p} are even, from (3.4) we get the following congruences

$$\sum_{i=1}^{t_{2p}} a_i + \sum_{i=1}^{t_p} 2k_i \equiv 0 (\operatorname{mod} 2p) \quad \text{and} \quad \sum_{i=1}^{t_2} (-1)^i j_i + \sum_{i=1}^{t_{2p}} (-1)^{t_2+i} f_i + \sum_{i=1}^{\gamma} \delta_i \equiv 0 (\operatorname{mod} m)$$

and when t_2, t_{2p} are odd, then we get

$$p + \sum_{i=1}^{t_{2p}} a_i + \sum_{i=1}^{t_p} 2k_i \equiv 0 (\operatorname{mod} 2p) \quad \text{and} \quad \sum_{i=1}^{t_2} (-1)^i j_i + \sum_{i=1}^{t_{2p}} (-1)^{t_2+i} f_i + \sum_{i=1}^{\gamma} \delta_i \equiv 0 (\operatorname{mod} m)$$

We can choose a_i etc. such that both the congruence's have solutions, so that ϕ is a smooth homomorphism.

We claim that when $\gamma = 0$, ϕ will be a smooth epimorphism only if $t_2 + t_{2p} \geq 4$.

Here $t_2 + t_{2p} \neq 1$ or 3, as $t_2 + t_{2p}$ is always even. When $t_2 + t_{2p} = 2$, the possibilities are $t_2 = t_{2p} = 1$; $t_2 = 2$, $t_{2p} = 0$ and $t_2 = 0$, $t_{2p} = 2$.

In all these three cases no finite order element of Γ is mapped to s and t simultaneously, satisfying the above two congruences and as a result $\phi(\Gamma) \subset G$. So ϕ is not onto. Hence if $\gamma = 0$, $\phi : \Gamma \to G$ will be a smooth epimorphism only if $t_2 + t_{2p} \geq 4$.

We can exhibit a smooth epimorphism $\phi : \Gamma \to G$ with these conditions as follows

$\phi(\alpha_i) = \phi(\beta_i) = 1$, $i = 1, 2, \cdots, \gamma$

$\phi(x_1) = t^p$, $\phi(x_i) = t^p s$, $i = 2, 3, \cdots, t_2$

$\phi(y_1) = t$, $\phi(y_2) = \phi(y_4) = \cdots = ts$; $\phi(y_3) = \phi(y_5) = \cdots = t^{2p-1}s$

$\phi(z_1) = \phi(z_3) = \cdots = t^{p-1}$, $\phi(z_2) = \phi(z_4) = \cdots = t^{p+1}$. (we take t_2, t_p, t_{2p} are odd).

Case III. When any three of the t_i's are zero:

(a) $t_2 = t_p = t_{m_j} = 0$. Here t_{2p} is even. From (3.4) we get

$$\sum_{i=1}^{t_{2p}} (-1)^i f_i + \sum_{i=1}^{t_{p_j}} h_i + \sum_{i=1}^{\gamma} \delta_i \equiv 0 (\mathrm{mod}\, m) \quad \text{and} \quad \sum_{i=1}^{t_{2p}} a_i + \sum_{i=1}^{t_{p_j}} b_i \equiv 0 (\mathrm{mod}\, 2p).$$

Both the congruences have solutions, if we choose the a_i etc. suitably and hence ϕ is a smooth homomorphism. And in view of (3.2), $\phi : \Gamma \to G$ will be a smooth epimorphism for $\gamma \geq 0$.

Similarly we have a smooth epimorphism $\phi : \Gamma \to G$ for $\gamma \geq 0$ in

(b) $t_2 = t_p = t_{p_j} = 0$, t_{2p} is even. (c) $t_2 = t_{m_j} = t_{p_j} = 0$, t_{2p} is even.

(d) $t_{2p} = t_p = t_{m_j} = 0$, t_2 is even and $t_{p_j} \geq 2$.

(e) $t_p = t_{m_j} = t_{p_j} = 0$, $t_2 + t_{2p}$ is even and $t_2 + t_{2p} \geq 4$ if $\gamma = 0$.

In the case (f) $t_2 = t_{2p} = t_p = 0$, we get from (3.4), $\sum_{i=1}^{t_{p_j}} b_i \equiv 0\,(\mathrm{mod}\, m)$, which implies $t_{p_j} \geq 2$ as $b_i < 2p$. If $\gamma = 0$, $\phi(\Gamma) \subset G$. So $\phi : \Gamma \to G$ will be a smooth epimorphism only if $\gamma \geq 1$.

We can show that the above conditions are also sufficient for the existence of a smooth epimorphism $\phi : \Gamma \to G$ defining ϕ as in (3.2).

In a similar way $\phi : \Gamma \to G$ will be a smooth epimorphism in the cases

(g) $t_2 = t_{2p} = t_{p_j} = 0$, if $t_p \geq 2$ and $\gamma \geq 1$. (h) $t_2 = t_{2p} = t_{m_j} = 0$, if $\gamma \geq 1$.

(i) $t_{2p} = t_p = t_{p_j} = 0$, t_2 is even, if $\gamma \geq 1$.

In case (j) $t_{2p} = t_{m_j} = t_{p_j} = 0$, t_2 is even, $\phi : \Gamma \to G$ will be a smooth homomorphism if $t_p \geq 2$.

Again from (3.2) it is clear that when $\gamma = 0$, $\phi : \Gamma \to G$ will be a smooth epimorphism if $t_2 \geq 4$ and when $\gamma \geq 1$, $t_2 \geq 2$.

Case IV. When any four of the periods are zero:

As in case II and III, $\phi : \Gamma \to G$ will be a smooth epimorphism in the following cases

(a) $t_2 = t_p = t_{m_j} = t_{p_j} = 0$, t_{2p} is even ≥ 2 and $\gamma \geq 0$

(b) $t_2 = t_{2p} = t_p = t_{p_j} = 0$, $\gamma \geq 1$. (c) $t_2 = t_{2p} = t_p = t_{m_j} = 0$, $t_{p_j} \geq 2$ and $\gamma \geq 1$.

(d) $t_{2p} = t_p = t_{m_j} = t_{p_j} = 0$, $t_2 \geq 2$ and $\gamma \geq 1$. (e) $t_2 = t_{2p} = t_{m_j} = t_{p_j} = 0$, $t_p \geq 2$ and $\gamma \geq 2$.

We now can conclude the above discussions in the form of a theorem.

Theorem 3.1. Let Γ be a Fuchsian group with presentation (1.1) and let G be the group with presentation (2.1) with $(p, m) = 1$. Then there exists a smooth epimorphism $\phi : \Gamma \to G$ if and only if

1. The periods of Γ (if any) take the values from the set $\{2, 2p, p, m_j, p_j\}$, $1 \leq j \leq m-1$, where $p_j = [p, m_j]$, $m_j = \dfrac{m}{d_j}$, $d_j = (m, j)$, $1 \leq j \leq m$.
2. If $t_2, t_{2p}, t_p, t_{m_j}, t_{p_j}$ denote the number of occurrences of periods $2, 2p, p, m_j, p_j$ respectively in Γ, then
   - (a) t_2 and t_{2p} are of same parity. If either of them is zero, other must be even.
   - (b) $\gamma = 0$ is possible in the following cases only
     - (i) when $t_{2p} \neq 0$, or
     - (ii) when $t_{2p} = 0$ but $t_2 \neq 0$ and at least one of t_p and t_{p_j} is nonzero. If $t_{p_j} = 0$, then $t_2 \geq 4$ for all t_p, or
     - (iii) when $t_{p_j} = t_{m_j} = 0$ and $t_2 + t_{2p} \geq 4$.

     Otherwise $\gamma \geq 1$. And $\gamma \geq 2$ when $t_2 = t_{2p} = t_{m_j} = t_{p_j} = 0$ and also when Γ is a surface group.
   - (c) (i) $t_{2p} \geq 2$, if $t_p = t_{p_j} = 0$,
     (ii) $t_p \geq 2$, if $t_{2p} = t_{p_j} = 0$,

(iii) when $t_{p_j} \geq 2$ if $t_{2p} = t_p = 0$.

Note. If $(p, m) \neq 1$, then $p = q_i$, where q_i is a factor of m, in the prime decomposition $m = q_1^{r_1} q_2^{r_2} q_3^{r_3} \cdots q_l^{r_l}$; $r_i > 0$; $q_1 < q_2 < \cdots < q_l$; $q_i \geq 3$. Therefore when $(p, m) \neq 1$, we get $p_j = m_j$ and in this case the periods of Γ will take the values from the set $\{2, 2p, m_j\}$ and we must have the following Theorem.

Theorem 3.2. Let Γ be a Fuchsian group with presentation (1.1) and let G be the group with presentation (2.1) and $(p, m) \neq 1$. Then there exists a smooth epimorphism $\phi : \Gamma \to G$ if and only if

1. The periods of Γ (if any) take the values from the set $\{2, 2p, m_j\}$, $m_j = \dfrac{m}{d_j}$, $d_j = h.c.f(m, j)$, $1 \leq j \leq m - 1$.
2. If t_2, t_{2p}, t_{m_j} denote the number of occurrences of periods $2, 2p, m_j$ respectively in Γ, then
   (a) t_2 and t_{2p} are of same parity. If either of them is zero, other must be even.
   (b) $\gamma = 0$ is possible in the following cases only
      (i) when $t_{2p} \neq 0$.If $t_{m_j} = 0$ then $t_{2p} \geq 2$ for all t_2, or
      (ii) when $t_{2p} = 0$ but $t_2 \neq 0$, $t_{m_j} \neq 0$ and $t_2 \geq 4$.

We now determine the minimum value of the genus $g \geq 2$ of the compact Riemann surface on which G acts as a group of automorphisms.

4. Determination of Minimum Genus

In Section 3 we have proved that the Fuchsian group Γ has the signatures,

$$\Delta\Big(\gamma, \underbrace{2, 2, \cdots, 2}_{t_2 \text{ times}}, \underbrace{2p, 2p, \cdots, 2p}_{t_{2p} \text{ times}}, \underbrace{p, p, \cdots, p}_{t_p \text{ times}}, \underbrace{m_j, m_j, \cdots, m_j}_{t_{m_j} \text{ times}}, \underbrace{p_j, p_j, \cdots, p_j}_{t_{p_j} \text{ times}}\Big), \text{ if } (p, m) = 1$$

and

$$\Delta\Big(\gamma, \underbrace{2, 2, \cdots, 2}_{t_2 \text{ times}}, \underbrace{2p, 2p, \cdots, 2p}_{t_{2p} \text{ times}}, \underbrace{m_j, m_j, \cdots, m_j}_{t_{m_j} \text{ times}}\Big), \text{ if } (p, m) \neq 1$$

where γ, t_i, m_j, p_j are as defined in Section 2 and Section 3 and satisfy the conditions of Theorem 3.1 and Theorem 3.2.

We can write

$$g = 1+pm\Big[2(\gamma-1)+(1-\frac{1}{2})t_2+(1-\frac{1}{2p})t_{2p}+\sum_{i=1}^{t_{m_j}}(1-\frac{1}{m_{j_i}})t_{m_{j_i}}+\sum_{i=1}^{t_{p_j}}(1-\frac{1}{p_{j_i}})t_{p_{ji}}\Big]$$

when $(p, m) = 1$ and

$$g = 1+pm\Big[2(\gamma-1)+\frac{t_2}{2}+\frac{2p-1}{2p}\,t_{2p}+\sum_{i=1}^{t_{m_j}}(1-\frac{1}{m_{j_i}})t_{m_{j_i}}\Big] \text{ when } (p,m)\neq 1.$$

We now determine the sets of values of γ and t_i's for which we may obtain the minimum values of g. We consider the prime decomposition of m as

$$m \;=\; q_1^{r_1}q_2^{r_2}q_3^{r_3}\cdots q_l^{r_l};\quad r_i>0,\; q_1<q_2<\cdots<q_l, q_i\geq 3. \tag{4.1}$$

Case 1. $\sum t_i = 0$. Here $\gamma \geq 2$. Therefore $g \geq 1 + 2pm$.

Case 2. $\sum t_i = 1$. Here $\gamma \geq 1$ and only possible period is m_j and $t_{m_j} = 1$.

Here $g \geq 1+pm\dfrac{m_j-1}{m_j} = 1+m\left(p-\dfrac{p}{m_j}\right)$.

The right hand expression will be minimum when m_j is minimum. But $m_j = \dfrac{m}{d_j}$, d_j=h.c.f.(j, m). Using the prime decomposition (4.1) for m, it can be verified that m_j is minimum when d_j is maximum, i.e, $d_j = q_1^{r_1-1}q_2^{r_2}q_3^{r_3}\cdots q_l^{r_l}$. So the minimum $m_j = q_1$.

Thus we get, $g \geq 1+m\left(p-\dfrac{p}{q_1}\right)$, $q_1 \geq 3$.

Case 3. $\sum t_i = 2$, $\gamma \geq 1$. g will be minimum when $t_2 = 2$ and $g \geq 1 + pm.1 = 1 + pm$.

Case 4. $\sum t_i \geq 3$, $\gamma \geq 0$. From (1.6) we have

$$g \;=\; 1+pm\left[2(\gamma-1)+\sum t_i-\left(\frac{t_2}{2}+\frac{t_{2p}}{2p}+\frac{t_p}{p}+\sum\frac{t_{m_j}}{m_j}+\sum\frac{t_{p_j}}{p_j}\right)\right].$$

Putting $\gamma \geq 1$ and $\sum t_i = 3$, we get

$$g \;\geq\; 1+pm\left[3-\left\{\frac{t_2}{2}+\frac{t_{2p}}{2p}+\frac{t_p}{p}+\sum\frac{t_{m_j}}{m_j}+\sum\frac{t_{p_j}}{p_j}\right\}\right].$$

As $\sum t_i = 3$ and $m, p \geq 3$, it can be verified that $\dfrac{t_2}{2} + \dfrac{t_{2p}}{2p} + \dfrac{t_p}{p} + \sum \dfrac{t_{m_j}}{m_j} + \sum \dfrac{t_{p_j}}{p_j} < 2$ and therefore $g > 1 + pm$.

Now we consider the cases for $\gamma = 0$, we have

$$g = 1 + pm\left[2(\gamma - 1) + \sum t_i - \left(\frac{t_2}{2} + \frac{t_{2p}}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_j}}{p_j}\right)\right].$$

Putting $\gamma = 0$ and $\sum t_i = 3$ we get

$$\begin{aligned} g = & \ 1 + pm\left[-2 + 3 - \left\{\frac{t_2}{2} + \frac{t_{2p}}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_j}}{p_j}\right\}\right] \\ = & \ 1 + pm\left[1 - \left\{\frac{t_2}{2} + \frac{t_{2p}}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_j}}{p_j}\right\}\right]. \end{aligned}$$

For $\sum t_i = 3$, the possible subcases are,

4(a) $t_2 = 1$, $t_{2p} = 1$, $t_{p_j} = 1$, $1 \leq j \leq m - 1$, $(p, m) = 1$.

4(b) $t_{2p} = 2$, $t_{p_j} = 1$, $1 \leq j \leq m - 1$, $(p, m) = 1$.

4(c) $t_{2p} = 2$, $t_{m_j} = 1$, $1 \leq j \leq m - 1$, $(p, m) = 1$.

4(d) $t_2 = 1$, $t_{2p} = 1$, $t_{m_j} = 1$, $1 \leq j \leq m - 1$, $(p, m) \neq 1$.

4(e) $t_{2p} = 2$, $t_{m_j} = 1$, $1 \leq j \leq m - 1$, $(p, m) \neq 1$.

By simple arithmetical calculations, it is observed that when $(p, m) = 1$, the minimum genus is obtained when the periods of Γ satisfy 4(a) and when $(p, m) \neq 1$, it is obtained when the periods satisfy 4(d) We determine the values of genus g for these two cases.

4(a) $t_2 = 1$, $t_{2p} = 1$, $t_{p_j} = 1$, $1 \leq j \leq m - 1$, $(p, m) = 1$.

$$\begin{aligned} g = 1 + pm\left\{1 - \left(\frac{1}{2} + \frac{1}{2p} + \frac{1}{p_j}\right)\right\} &= 1 - pm\,\frac{p+1}{2p} + pm\left(1 - \frac{1}{p_j}\right) \\ = \ 1 - m\,\frac{p+1}{2} + m\left(p - \frac{p}{p_j}\right) & \qquad (4.2) \end{aligned}$$

g will be minimum if p_j is minimum. Using prime decomposition (4.1) of m, g is minimum when $p_j = l.c.m.[p, q_1] = pq_1$ as $(p, m) = 1$.

When $p = 3$,

$$g = 1 + m\left(1 - \frac{p}{p_j}\right) \qquad (4.3)$$

4(d) $t_2 = 1$, $t_{2p} = 1$, $t_{m_j} = 1$, $1 \le j \le m-1$, $(p,m) \ne 1$. Here

$$g = 1 + pm\left\{1 - \left(\frac{1}{2} + \frac{1}{2p} + \frac{1}{m_j}\right)\right\} = 1 - \frac{p+1}{2}m + m\left(p - \frac{p}{m_j}\right) \qquad (4.4)$$

Taking the prime decomposition (4.1) of m, the minimum g is $g = 1 - m\frac{p+1}{2} + m\left(p - \frac{p}{q_i}\right)$.

When $p = m = 3$, the condition of 4(d) is untenable for $\gamma = 0$, by (1,3). In this special case the minimum genus is obtained when $t_{2p} = 2$, $t_{m_j} = 1$. Here

$$g = 1 + pm\left\{1 - \left(\frac{2}{2p} + \frac{1}{m_j}\right)\right\} = 1 + 3.3\left\{1 - \left(\frac{2}{6} + \frac{1}{3}\right)\right\} = 4.$$

We get the minimum genus g for $\sum t_i = 3$, as follows

$$g = 1 - m\frac{p+1}{2} + m\left(p - \frac{p}{pq_1}\right), \quad (p,m) = 1;$$

$$g = 1 - m\frac{p+1}{2} + m\left(p - \frac{p}{q_1}\right), \quad (p,m) \ne 1;$$

and $g = 4$, when $p = m = 3$. In these cases the corresponding Fuchsian groups have the signatures $\Delta(0; 2, 2p, p_j)$, $\Delta(0; 2, 2p, m_j)$ and $\Delta(0; 3, 6, 6)$ respectively and $p_j = pq_1$ and $m_j = q_1$ by (4.1).

We now show that g cannot have a lower minimum for $\sum t_i \ge 4$. It can be verified that if g takes a lower minimum value for $\sum t_i \ge 4$, then it is obtained for $\gamma = 0$ only.

Putting $\gamma = 0$ in (1.6) we have

$$g = 1 + pm\left[-2 + \sum t_i - \left(\frac{t_2}{2} + \frac{t_{2p}}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_j}}{p_j}\right)\right].$$

Considering $\sum t_i = 4$, we get the possible values of the periods of Γ as:

(i) $t_2 = t_{2p} = 2$; (ii) $t_2 = t_{2p} = 1$, $t_{p_j} = 2$; (iii) $t_{2p} = 2$, $t_{p_j} = 2$; (iv) $t_{2p} = 4$.

With simple arithmetical calculations we can say that the minimum g is obtained when $t_2 = t_{2p} = 2$ and then

$$g \;=\; 1 + pm\left[1 - \frac{1}{p}\right]. \qquad (4.5)$$

Now suppose if possible

$$1 + pm\left[1 - \frac{1}{p}\right] < 1 - \frac{p+1}{2}m + m\left(p - \frac{p}{pq_1}\right).$$

We have

$$1+pm\left[1-\frac{1}{p}\right]-\left[1-\frac{p+1}{2}m+m\left(p-\frac{p}{pq_1}\right)\right]=m\left[\frac{p+1}{2}-1+\frac{p}{pq_1}\right]>0,$$

a contradiction.

Therefore the value of g obtained in (4.5) is not smaller than that obtained in (4.2) for $(p,m)=1$. Similarly it can be shown that when $(p,m)\neq 1$, it is not smaller than that of (4.3). If $p=m=3$, $g=7$ when $t_2=t_{2p}=2$ and it is also greater than 4.

The case $\sum t_i \geq 5$ can be disposed of similarly. The above discussions may be concluded as

Theorem 4.1. The minimum value of the genus g of a compact Riemann surface having G, as it's group of automorphisms is

(i) $g=1-m\dfrac{p+1}{2}+m\left(p-\dfrac{p}{pq_1}\right)$, if $(p,m)=1$;

(ii) $g=1-m\dfrac{p+1}{2}+m\left(p-\dfrac{p}{q_1}\right)$, if $(p,m)\neq 1$;

(iii) $g=4$, if $p=m=3$,

where $m=q_1^{r_1}q_2^{r_2}q_3^{r_3}\cdots q_l^{r_l}$; $r_1>0$, $q_1<q_2<\cdots<q_l$, $q_i\leq 3$.

The results of Harvey and of Chutiya can be deduced as a corollary of our Theorem 3.1, Theorem 3.2 and Theorem 4.1.

Corollary 4.1. Considering a cyclic group C_p, p is an odd prime, as $C_p\cong\bar{G}$, $\bar{G}=I_1\times C_p$ where I_1 is the identity group of D_m, a dihedral group, we obtain the result as follows.

A smooth epimorphism $\phi:\Gamma\to C_p$, p is an odd prime, will exist if and only if

1. $\gamma\geq 2$ in Γ, if Γ is a surface group.
2. p is the only period of Γ.
3. if t_p denotes the number of occurrences of period p, then
   (i) $\gamma=0$ is possible if $t_p\geq 3$ for $p\neq 3$,
   (ii) $t_p\geq 4$ when $\gamma=0$ and $p=3$.

The minimum genus g of the compact Riemann surface on which C_p acts as a group of automorphisms, is

(i) $g = 2$ when $p = 3$.

(ii) $g = 1 + \dfrac{p-3}{2}$ when $p > 3$.

$\Delta(0; 3, 3, 3, 3)$ and $\Delta(0; p, p, p)$ are the signatures of the respective Fuchsian groups.

Corollary 4.2. Considering a dihedral group D_m, m is an odd integer greater than one, as $D_m \cong \bar{D}_m, \bar{D}_m = D_m \times I_2$, I_2 is the identity group of C_p, we obtain a result as follows.

A smooth epimorphism $\phi : \Gamma \to D_m$, m is an odd integer greater than one, exists if and only if

1. $\gamma \geq 2$, if Γ is a surface group,
2. Γ takes periods (if any) from $\{2, m_j\}$, $m_j = \dfrac{m}{d_j}$, $d_j = (m, j)$ $1 \leq j \leq m$.
3. if t_2 denotes the number of occurrences of period 2, then
   (i) $\gamma = 0$ is possible if $t_2 \neq 0$ otherwise $\gamma \geq 1$.
   (ii) $t_2 \geq 2$, and always even.

The minimum genus g of the compact Riemann surface on which D_m acts as a group of automorphisms is $g = 1 + m\left(1 - \dfrac{2}{q_1}\right)$ accordiog to the prime decomposition of m given in (4.1) and the corresponding Fuchsian group has the signature $\Delta(0; 2, 2, q_1, q_1)$.

References

[1] Burnside, *The Theory of Groups of Finite Order (Note K)*, Dover 1995.

[2] Chetiya, B.P., *On genuses of compact Riemann surfaces admitting solvable automorphism groups*, Indian J. Pure and Appl. Math. **12** (1991).

[3] Chetiya, B.P. and Dutta, S. *Some triangular groups as automorphism groups of compact Riemann surfaces*, Far East J. Math. Sci. **2**(3) (2000),441–451.

[4] Chetiya, B.P., Dutta S.K. and Patra, K., *Genera of compact Riemann surfaces admitting Dihedral automorphism groups*, J. Ramanujan Math. Soc. **11**(2) (1996), 127–138.

[5] Chetiya, B.P., Dutta S.K. and Patra, K., *On ZS-metacyclic groups of automorphisms of compact Riemann surfaces*, Indian J. Pure and Appl. Math. **28**(1) (1997), 63–74.

[6] Chetiya, B.P. and Patra, K., *K-metacyclic groups of automorphisms of compact Riemann surfaces*, Far East J. Math Soc. **202** (1994), 126–137.

[7] Chutia, C., *Solution to Some Problems on Riemann Surface Automorphism Groups*, Ph.D. Thesis (G.U), 1998.

[8] Coxeter, H.S.M. and Moser, W.O.J., *Generators and Relations for Discrete Groups*, Springer-Verlag, 1972.

[9] Glover, H. and Sjerve, D., *Representing PSL2(p) on a Riemann surface of least genus*, L. Enscign Math. **31** (1985), 305–325.

[10] Harvey, W.J., *Cyclic groups of automorphisms of compact Riemann surfaces*, Quart. J. Math. Oxford. **17**(2) (1966), 86–97.

[11] Hurwitz, A., *Ueber algebraische gebilde mit eindeutigen Transformationen in sich* (Germen), Math. Ann. **41** (1892), 403–442.

[12] Machbeath, A.M., *Discontinuous groups and birational transformations*, Proc. of Dundee Summer School, Univ. of St. Andrews, 1961.

[13] Machbeath, A.M., *On a theorem of Hurwitz*, Proc. Glasgow Math. Asser. **5** (1961) 90–96.

[14] Maclachlan, C., *Abelian groups of automorphisms of compact Riemann surfaces*, Proc. London Math. Soc. **15** (1965), 699–712.

[15] Maclachlan, C., *Topics on Riemann surface and Fuchsian groups*, London Math. Soc. Lecturer Note Ser. **287** Cambrige Univ. Press, Cambridge, 2001.

[16] Michael, A.A., *Metacyclic groups of automorphisms of compact Riemann surfaces*, Hiroshima Math. J. **31**(1) (2000), 117–132.