South East Asian J. Math. & Math. Sc.
Vol.6 No.1(2007), pp.11–16

ON THE (P) SUMMABILITY OF DOUBLE FOURIER SERIES

Satish Chandra and Manisha Sharma*

Department of Mathematics
S.M. Post-Graduate College, Chandausi-202412, India

*Indian Institute of Management, Ahmedabad-380015, India

(*Received: October 30, 2004*)

Abstract: In this paper, we have proved a theorem on (P) summability of double Fourier series which generalizes various known results.

Keywords and Phrases: (P) summability, double Fourier series
2000 AMS Subject Classification: 40C05, 40D05, 40E05, 40F05, 40G05

1. Definitions and Notations

Let $\{S_{m,n}\}$ be the sequence of mn^{th} partial sums of the series $\sum u_{m,n}$. Let $\{p_m\}$ and $\{q_n\}$ be sequences of non-negative numbers such that the series

$$p(x,y) \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_m q_n x^m y^n \tag{1.1}$$

converges for all x and y, $0 < x < 1$, $0 < y < 1$ and $p(x,y) \uparrow \infty$ as $x \uparrow 1$ and $y \uparrow 1$.

If

$$p(x,y) \;=\; \frac{1}{p(x,y)} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} s_{m,n} p_m q_n x^m y^n \to S \tag{1.2}$$

as $x \uparrow 1$ and $y \uparrow 1$, then the series $\sum u_{m,n}$ is said to be (P)-summable to S [5].

If

$$L(x,y) \;=\; \frac{1}{|\log(1-x)|\,|\log(1-y)|} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{s_{m,n} x^m y^n}{mn} \to S \tag{1.3}$$

as $x \uparrow 1$ and $y \uparrow 1$, then the series $\sum u_{mn}$ is said to be L-summable to S [8].

In particular if $p_m = \dfrac{1}{m}$ and $q_n = \dfrac{1}{n}$, the (P)-summability reduces to (L) summability.

Let $f(x,y)$ be a periodic function with period 2π in each variable and integrable in the sense of Lebesgue over the square $S(-\pi,\pi;-\pi,\pi)$.

Let the series

$$\sum_{m,n=0}^{\infty} \lambda_{m,n}(a_{m,n}\cos mx\cos ny + b_{m,n}\sin mx\cos ny + c_{m,n}\cos mx\sin ny$$
$$+d_{m,n}\sin mx\sin ny), \tag{1.4}$$

be called the double Fourier series of the function $f(x,y)$ where $\lambda_{m,n}$ are given by

$$\lambda_{m,n} = \begin{cases} \frac{1}{4} & \text{for } m=0,\ n=0 \\ \frac{1}{2} & \text{for } m=0,\ n>0;\ m>0,\ n=0 \\ 1 & \text{for } m>0,\ n>0 \end{cases} \tag{1.5}$$

and the coefficient in (1.4) are given by

$$a_{m,n} = \frac{1}{\pi^2}\int\int_S f(u,v)\cos mu\cos nv du\, dv \tag{1.6}$$

and three similar expressions defining $b_{m,n}$, $c_{m,n}$ and $d_{m,n}$. Let us write

$$\phi(u,v) = [f(x+u,y+v)+f(x+u,y-v)+f(x-u,y+v)+f(x-u,y-v)-4f(x,y)].$$

Let

$$p'(x,y) = \frac{\partial^2}{\partial x\partial y}p(x,y), \quad H(x,u) = \int_0^u L(x,y)\,dy, \quad H(y,v) = \int_0^u L(x,y)\,dx$$

and $H(x,\pi)$ is $H(x,u)$ at $u=\pi$.

2. Results

Concerning (P) summability of a differentiated Fourier series, Hotta [5] has proved the following theorem:

Theorem 2.1. If $\{np_n\}$ is a monotonic convex sequence such that

$$\frac{p(x)}{(1-x)^2\ p'(x)} \to \infty \quad \text{and} \quad \int_{1-x}^{\pi} \frac{H(t)}{t^3}\,dt = o\left(\frac{p(x)}{(1-x)^2\ p'(x)}\right)$$

as $x \uparrow 1$, then the series $\sum_{n=1}^{\infty} nB_n(t)$ is summable (P) to C at $t=x$.

The aim of this paper is to generalize Theorem 2.1 for summability of double Fourier series.

We shall prove the following theorem:

Theorem 2.2. If $\{mp_n\}$ and $\{nq_n\}$ are the monotonic convex sequences such that

$$\frac{p(x,y)}{(1-x)^2\ (1-y)^2\ p'(x,y)} \to \infty, \tag{2.1}$$

and

$$\begin{aligned}\Phi \ &= \ \int_x^\pi \int_y^\pi \frac{\phi(u,v)}{uv}\, du\, dv \\ &= \ o\left[\frac{p(x,y)}{(1-x)^2\ (1-y)^2\ p'(x,y)}\right]\end{aligned} \tag{2.2}$$

as $x \to +\infty$ and $y \to +\infty$, then the double Fourier series (1.4) is summable (P) to $f(x,y)$ at the point $(u,v) = (x,y)$.

Proof: Let $S_{m,n}$ denote the $(m,n)^{\text{th}}$ partial sum of the series (1.4), then we know that

$$\begin{aligned}S_{m,n} - f(x,y) &= \frac{1}{4\pi^2}\int_0^\pi \int_0^\pi \phi(u,v)\, \frac{\sin(m+1/2)u \sin(n+1/2)v}{\sin\frac{1}{2}\,u\ \sin\frac{1}{2}\,v}\, du\, dv \\ &= \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \phi(u,v)\, \frac{\sin mu\ \sin nv}{u\, v}\, du\, dv + o(1)\\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |s_{m,n} - f(x,y)| \frac{x^m y^n}{mn} &= \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \frac{\phi(u,v)}{uv} \sum_{m=1}^{\infty} \frac{x^m \sin mu}{m} \sum_{n=1}^{\infty} \frac{y^n \sin nv}{n} dudv \\ &\quad + o\big((1-x)^2(1-y)^2\big) \\ &= \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \frac{\phi(u,v)}{uv}\, H(x,u)H(y,v) du\, dv \\ &\quad + o\big((1-x)^2(1-y)^2\big) \\ &= I + o\big((1-x)^2(1-y)^2\big)\end{aligned}$$

where $H(x,u) = \tan^{-1}\left\{\dfrac{x \sin u}{1 - x\cos u}\right\}$.

Thus it suffices for our purpose to show that

$$\begin{aligned}I = \ &\frac{1}{\pi^2}\int_0^\pi \int_0^\pi \frac{\phi(u,v)}{uv}\, H(x,u)H(y,v) du\, dv \\ = \ &o\big((1-x)^2(1-y)^2\big),\end{aligned}$$

as $x \to 1-0$ and $y \to 1-0$.

Now, let $\xi = (1-x)$ and $\eta = (1-y)$, let us set

$$\pi^2 I = \left(\int_0^\xi \int_0^\eta + \int_0^\xi \int_\eta^\pi + \int_\xi^\pi \int_0^\eta + \int_\xi^\pi \int_\eta^\pi\right) \frac{\phi(u,v)}{uv} H(x,u)H(y,v)dudv \tag{2.3}$$
$$= I_1 + I_2 + I_3 + I_4.$$

We have the following estimates

$$\begin{cases} H(x,u) = o(u/\xi), & \text{for } 0 \le u \le \xi, \\ H(x,\pi) = 0, & \text{and } H(x,\xi) = o(1), \end{cases} \tag{2.4}$$

uniformly for $0 \le u \le \pi$ and $0 \le x < 1$.

Also

$$\frac{dH(x,u)}{du} = H'(x,u) = \frac{(\cos u - x)}{1 - 2x\cos u + x^2}$$
$$= \begin{cases} o\left[\dfrac{1}{(1-x)}\right] = o(1/\xi), & \text{for } u \le \xi, \\ o\left[\dfrac{(1-x)}{u^2}\right] = o(\xi/\eta^2), & \text{for } u > \xi, \end{cases} \tag{2.5}$$

Since

$$\frac{\phi(u,v)}{uv} = \frac{d^2\phi(u,v)}{dudv}$$

almost everywhere therefore, by (2.2),(2.4) and (2.5), we have

$$I_1 = \int_0^\xi \int_0^\eta \frac{d^2\phi(u,v)}{dudv} H(x,u)H(y,v)dudv$$
$$= \left[\phi(u,v)H(x,u)H(y,v) - \int \phi(u,v)\frac{dH(x,u)}{du}H(y,v)\,du \right.$$
$$\left. - \int \phi(u,v)H(x,u)\frac{dH(y,v)}{dv}\,dv + \int\int \phi(u,y)\,\frac{d^2H(x,u)H(y,v)}{du\,dv}\right]_{0,0}^{\xi,\eta}$$
$$= o\left[\left\{\frac{1}{(1-u)^2}\cdot\frac{1}{(1-v)^2}\,\frac{uv}{\xi\eta}\right\}_{0,0}^{\xi,\eta}\right] + o\left[\left\{\int \frac{1}{(1-u)^2}\cdot\frac{1}{(1-v)^2}\,\frac{v}{\xi\eta}\,du\right\}_{0,0}^{\xi,\eta}\right]$$
$$+ o\left[\left\{\int \frac{1}{(1-u)^2}\cdot\frac{1}{(1-v)^2}\,\frac{u}{\xi\eta}\,dv\right\}_{0,0}^{\xi,\eta}\right] + o\left[\int_0^\xi \int_0^\eta \frac{\frac{1}{(1-u)^2}\cdot\frac{1}{(1-v)^2}}{\xi\eta}\,dudv\right]$$
$$= o\left[\frac{1}{(1-\xi)^2}\cdot\frac{1}{(1-\eta)^2}\right]$$

Since $$\int_0^{\xi} \frac{1}{(1-u)^2}\,du = \frac{1}{(1-\xi)}.$$

Again by (2.5), we have

$$\begin{aligned}
I_2 = &\int_0^{\xi}\int_0^{\pi} \frac{d^2\phi(u,v)}{dudv}\,H(x,u)H(y,v)dudv \\
= &\left\{\phi(u,v)H(x,u)H(y,v) - \int \phi(u,v)\frac{dH(x,u)}{du}\,H(y,v)\,du\right. \\
&\left. - \int \phi(u,v)H(x,u)\frac{dH(y,v)}{dv}\,du + \int\int \phi(u,v)\,\frac{dH(x,u)H(y,v)}{du\,dv}\,dudv\right\}_{0,\eta}^{\xi,\pi} \\
= &\; o\left[\frac{1}{(1-u)^2}\cdot\frac{1}{(1-v)^2}\,|H(y,v)|\,\frac{u(\xi,\pi)}{\xi(0,\eta)}\right] \\
&+ o\left[\left\{\int \frac{1}{(1-u)^2}\cdot\frac{1}{(1-v)^2}\,\frac{|H(y,v)|}{\xi}\,du\right\}_{0,\eta}^{\xi,\pi}\right] \\
&+ o\left[\left\{\int \frac{1}{(1-u)^2}\cdot\frac{1}{(1-v)^2}\,\frac{\eta u}{v\xi}\,dv\right\}_{0,\eta}^{\xi,\pi}\right] \\
&+ o\left[\int_0^{\xi}\int_{\eta}^{\pi} \frac{\frac{1}{(1-u)^2}\cdot\frac{1}{(1-v)^2}\,\eta\,dudv}{v\xi}\right] \\
= &\; o\left[\frac{1}{(1-\xi)^2}\cdot\frac{1}{(1-\eta)^2}\right] + o\left[\frac{1}{\xi}\,\frac{1}{(1-\eta)^2}\int_0^{\xi}\frac{1}{(1-u)^2}\,du\right] \\
= &\; o\left[\eta\frac{1}{(1-\xi)^2}\int_{\eta}^{\pi}\frac{\frac{1}{(1-v)^2}}{v^2}\,dv\right] + o\left[\frac{\eta}{\xi}\int_0^{\xi}\frac{1}{(1-u)^2}\,du\int_{\eta}^{\pi}\frac{(1-v)^2}{v^2}\,dv\right] \\
= &\; o\left[\frac{1}{(1-\xi)^2}\cdot\frac{1}{(1-\eta)^2}\right]
\end{aligned}$$

Similarly $$I_3 = o\left[\frac{1}{(1-\xi)^2}\cdot\frac{1}{(1-\eta)^2}\right]$$

as above

$$\begin{aligned}
I_4 = &\left\{\phi(u,v)H(x,u)H(y,v) - \int \phi(u,v)\frac{dH(x,u)}{du}\,H(y,v)\,du\right. \\
&\left. - \int \phi(u,v)\frac{dH(y,v)}{dv}\,H(x,u)dv + \int\int \phi(u,v)\,\frac{d^2H(x,u)H(y,v)}{du\,dv}\,dudv\right\}_{\xi,\eta}^{\pi,\pi}
\end{aligned}$$

$$
\begin{aligned}
&= o\left[\left\{\frac{1}{(1-u)^2}\cdot\frac{1}{(1-v)^2}\,|H(x,u)|\,|H(y,v)|\right\}_{\xi,\eta}^{\pi,\pi}\right] \\
&\quad + o\left[\frac{1}{(1-\eta)^2}\int_\xi^\pi \frac{1}{(1-u)^2}\,\frac{\xi}{u^2}\,du\right] + o\left[\frac{1}{(1-\xi)^2}\int_\eta^\pi \frac{1}{(1-v)^2}\,\frac{\eta}{v^2}\,dv\right] \\
&\quad + o\left[\xi\eta\int_\xi^\pi \frac{du}{(1-u)^2u^2}\int_\eta^\pi \frac{dv}{(1-v)^2v^2}\right] \\
&= o\left[\frac{1}{(1-\xi)^2}\cdot\frac{1}{(1-\eta)^2}\right]
\end{aligned}
$$

This completes the proof.

References

[1] Alexit, G. and Turin, P., *Fourier Analysis and Approximation Theory*, Vols. I and II, North Holland Publishing Company, Amsterdam, Oxford, New York, 1978.

[2] Borwein, D., *A logarithmic method of summability*, J. London Math. Soc. **33** (1958), 212–220.

[3] Chandra, S., *On (N, p_m, q_n) summability of a double Fourier series*, The Aligarh Bull. of Math. **21**(1) (2002), 55–60.

[4] Hardy, G.H., *Divergent Series*, Oxford University Press, Oxford, 1949.

[5] Hotta, G.C., *On the (P) summability of a differential Fourier series*, Bull. Cal. Math. Soc. **73** (1981), 205–210.

[6] Hsiang, F.C., *Summability (L) of the Fourier series*, Bull. Amer. Math. Soc. **67** (1961), 150–153.

[7] Islam, R.M. and Maniruzzaman, M., *On $K^{\lambda uv}$-summability of a triple Fourier series*, Bull. Cal. Math. Soc. **91**(4) (1999), 323–332.

[8] Mohanty, R. and Nanda, M., *The summability (L) of the differential Fourier series*, Quart. J. Math. (Oxford) **13**(2) (1962), 40–44.

[9] Nanda, M., *The summability (L) of Fourier series and first differential series*, Quart. J. Math. (Oxford) **13**(2) (1962), 229–234.

[10] Singh, N. and Sharma, N., *On product of double series*, Bull. Cal. Math. Soc. **93**(2) (2001), 145–154.

[11] Zygmund, A., *Trigonometric Series*, Vols. I and II (II Edition), Cambridge University Press, Cambridge, 1959.