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BAYESIAN ESTIMATION OF THE SHAPE PARAMETER OF FINITE RANGE DISTRIBUTION USING LINEX LOSS FUNCTION WITH TYPE II CENSORING

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Abstract: In this paper we have considered the Bayesian estimation under Type II for the shape parameter of the Finite Range distribution using linex loss function under quasi, natural conjugate and uniform prior distributions. These estimators are compared with the corresponding Bayes estimators under squared error loss function.

Keywords and Phrases: Squared error loss function, asymmetric loss function, prior distribution, diffuse and non informative prior, posterior pdf and expectation, inverted gamma distribution
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1. Introduction

Let us consider the Finite Range distribution with probability density function (pdf) given by

$$f(x;\sigma,\theta) \;=\; \frac{1}{\theta x}\left(\frac{x}{\sigma}\right)^{\frac{1}{\theta}}; \quad \theta > 0,\ \sigma > 0,\ 0 < x \leq \sigma, \tag{1.1}$$

where σ is known (Mukherjee and Islam [4]). Let us suppose that n items, having the life time distribution with *pdf* as (1.1), are put to life test experiment, without replacement, and the experiment is terminated as soon as $r(\leq n)$ items have failed. If $\underline{X} = (X_1, \cdots, X_r)$ denote the random vector of the r observations (life times) as obtained above. The joint pdf of $\underline{X}$ is given by

$$f(\underline{x}|\theta) \;=\; \frac{n!}{(n-r)!}\left(\frac{1}{\theta}\right)^r \left(\prod_{i=1}^{r} x_i\right) e^{-\left(\frac{T_r}{\theta}\right)}, \tag{1.2}$$

where

$$T_r \;=\; \left[\sum_{i=1}^{r} \log\left(\frac{\sigma}{x_i}\right) + (n-r)\log\left(\frac{\sigma}{x_{(r)}}\right)\right].$$

Thus, the MLE of θ is given by

$$\hat{\theta} \;=\; \frac{T_r}{r} \tag{1.3}$$

and the *pdf* of $\hat{\theta}$ is given by

$$f(\hat{\theta}) \;=\; \frac{\left(\frac{r}{\theta}\right)^r}{\Gamma(r)}\,(\hat{\theta})^{r-1}\,e^{-r\hat{\theta}/\theta}; \qquad \hat{\theta} > 0. \tag{1.4}$$

The Bayes estimator $\hat{\theta}_L$ of θ, of course, is the optimal estimator relative to the chosen loss function L. A commonly used loss function is the squared error loss function (SELF), is symmetric and is given as

$$L(\hat{\theta}, \theta) \;=\; (\hat{\theta} - \theta)^2. \tag{1.5}$$

It is well known that the Bayes estimator under the above loss function, say $\hat{\theta}_S$, is the posterior mean. The SELF is often used due to the fact that it is symmetrical and also that it does not lead to complicated numerical methods for various calculations. Several authors, viz., Ferguson [3], Varian [6], Berger [2], Zellner [7] and Basu and Ebrahimi [1], to name a few, have recognized the inappropriateness of using symmetric loss functions in several estimation problems. These authors have proposed different asymmetric loss functions e.g., Linex and many of it's variant forms.

Varian [6] introduced the following convex loss function known is LINEX (linear-exponential) loss function

$$L(\Delta) \;=\; be^{a\Delta} - c\Delta - b; \qquad a, c \neq 0, \quad b > 0, \tag{1.6}$$

where $\Delta = \hat{\theta} - \theta$. It is clear that $L(0) = 0$ and the minimum occurs when $ab = c$, therefore, $L(\Delta)$ can be written as

$$L(\Delta) \;=\; b\left[e^{a\Delta} - a\Delta - 1\right], \qquad a \neq 0, \quad b > 0, \tag{1.7}$$

where a and b are the parameters of the loss function may be defined as shape and scale respectively. This loss function has been considered by Zellner [7], Rojo [5]. Basu and Ebrahimi [1] considered the $L(\Delta)$ as

$$L(\Delta) \;=\; b\left[e^{a\Delta} - a\Delta - 1\right], \qquad a \neq 0, \quad b > 0, \tag{1.8}$$

where

$$\Delta \;=\; \frac{\hat{\theta}}{\theta} - 1$$

and studied the Bayesian estimation under this asymmetric loss function for exponential lifetime distribution. This loss function is suitable for the situations where overestimation of θ is more costly than its underestimation.

Thus Bayes estimator under asymmetric loss $L(\Delta)$, i.e., a $\hat{\theta}_A$ is the solution of the following equation

$$E_\pi \left[\frac{1}{\pi} \exp\left(\frac{a\hat{\theta}_A}{\theta} \right) \right] \;=\; e^a E_\pi \left(\frac{1}{\theta} \right) \tag{1.9}$$

In this paper, we have obtained Bayes estimator of θ using linex loss function, under three prior distributions viz., quasi-density

$$g_1(\theta) \;=\; \frac{1}{\theta^d}; \qquad \theta > 0, \quad d \geq 0, \tag{1.10}$$

here $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior; the inverted gamma distribution as natural conjugate with parameters α and β (> 0) with p.d.f. given as

$$g_2(\theta) \;=\; \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}\, \theta^{-(\alpha+1)}\, e^{-\beta/\theta} & ; \;\; \theta > 0 \;\; (\alpha, \beta) > 0 \\ 0 & ; \;\; \text{otherwise} \end{cases} \tag{1.11}$$

and uniform prior over $[\alpha, \beta]$ as

$$g_3(\theta) \;=\; \begin{cases} \frac{1}{\beta - \alpha} & ; \;\; 0 < \alpha \leq \theta \leq \beta \\ 0 & ; \;\; \text{otherwise} \end{cases} \tag{1.12}$$

2. Bayes Estimator of θ under Quasi Prior $g_1(\theta)$

The posterior pdf of θ under $g_1(\theta)$, may be obtained, using equation (1.2),

$$f(\theta|\underline{x}) \;=\; \frac{T_r^{r+d-1}}{\Gamma(r+d-1)}\, \theta^{-(r+d)}\, e^{-(T_r)/\theta}, \quad \theta > 0, \; r + d > 1. \tag{2.1}$$

The Bayes estimator under SELF is given by

$$\hat{\theta}_S \;=\; \frac{T_r}{(r+d-2)}; \qquad r + d > 2. \tag{2.2}$$

Also, the Bayes estimator under linex loss function is obtained as

$$\hat{\theta}_A \;=\; \left(\frac{1 - e^{-a/(r+d)}}{a} \right) T_r. \tag{2.3}$$

The Risk Functions. The risk function of the estimators $\hat{\theta}_S$ and $\hat{\theta}_A$, relative to SELF are denoted by $R_S(\hat{\theta}_S)$ and $R_S(\hat{\theta}_A)$, respectively and those relative to linex $R_A(\hat{\theta}_S)$ and $R_A(\hat{\theta}_A)$, respectively are given by

$$R_S(\hat{\theta}_S) = \theta^2 \left[\frac{r(r+1)}{(r+d-2)^2} - \frac{2r}{(r+d-2)} + 1 \right], \tag{2.4}$$

$$R_S(\hat{\theta}_A) = \theta^2 \left[\frac{r(r+1)}{a^2}(1-e^{-a/(r+d)})^2 - \frac{2r}{a}(1-e^{-a/(r+d)}) + 1 \right], \tag{2.5}$$

$$R_A(\hat{\theta}_S) = b\left[e^{-a}\left(1 - \frac{a}{r+d-2}\right)^{-r} - \left(\frac{ar}{r+d-2}\right) + a - 1 \right], \tag{2.6}$$

$$R_A(\hat{\theta}_A) = b\left[e^{-ad/(r+d)} - r(1-e^{-a/(r+d)}) + a - 1 \right]. \tag{2.7}$$

3. Bayes Estimator of θ under Natural Conjugate Prior $g_2(\theta)$

The posterior pdf of θ under $g_2(\theta)$, using equation (1.2) comes out to be

$$f(\theta|\underline{x}) = \frac{(\beta+T_r)^{r+\alpha}}{\Gamma(r+\alpha)}\, \theta^{-(r+\alpha+1)}\, e^{-\frac{1}{\theta}(\beta+T_r)}. \tag{3.1}$$

Using equation (3.1), the Bayes estimator under SELF is given by

$$\hat{\theta}_S = \frac{\beta+T_r}{(r+\alpha-1)}; \qquad r+d>2. \tag{3.2}$$

The Bayes estimator under linex loss function $L(\Delta)$, using the value of $f(\theta|\underline{x})$ from equation (3.1) is the solution of equation (1.9) given by

$$\hat{\theta}_A = \left(\frac{1-e^{-a/(r+\alpha+1)}}{a} \right)(\beta+T_r). \tag{3.3}$$

The Risk Functions. The risk function of the estimators $\hat{\theta}_S$ and $\hat{\theta}_A$, relative to SELF are given by

$$R_S(\hat{\theta}_S) = \theta^2 \left[\left(\frac{r(r+1) + \frac{2r\beta}{\theta} + \frac{\beta^2}{\theta^2}}{(r+\alpha-1)^2} \right) - \frac{2\left(r+\frac{\beta}{\theta}\right)}{(r+\alpha-1)} + 1 \right], \tag{3.4}$$

and

$$R_S(\hat{\theta}_A) = \theta^2 \left[C^2 \left(r(r+1) + \frac{2r\beta}{\theta} + \frac{\beta^2}{\theta^2} \right) - 2C\left(r+\frac{\beta}{\theta}\right) + 1 \right], \tag{3.5}$$

where

$$C = \left(\frac{1-e^{-a/(r+\alpha+1)}}{a}\right).$$

The risk functions of the estimators $\hat{\theta}_S$ and $\hat{\theta}_A$, relative to linex loss are given by

$$R_A(\hat{\theta}_S) = b\left[\left(e^{-a\left(1-\frac{\beta}{\theta(r+\alpha-1)}\right)}\right)\left(1-\frac{a}{r+\alpha-1}\right)^{-r} - \left(\frac{a\left(r+\frac{\beta}{\theta}\right)}{r+\alpha-1}\right) + a - 1\right] \tag{3.6}$$

and

$$R_A(\hat{\theta}_A) = b\left[\left(e^{\frac{-a(\alpha+1)}{(r+\alpha+1)}}\right)\left(e^{\frac{\beta}{\theta}(1-e^{a/(r+\alpha-1)}}\right) - \left(1-e^{\frac{-a}{(r+\alpha+1)}}\right)\left(r+\frac{\beta}{\theta}\right) + a - 1\right]. \tag{3.7}$$

4. Bayes Estimator of θ under Uniform Prior $g_3(\theta)$

The posterior pdf under $g_3(\theta)$ may be obtained as

$$f(\theta|\underline{x}) = \frac{T_r^{r-1}\theta^{-r}e^{-T_r/\theta}}{I_g\left(\frac{T_r}{\alpha}, r-1\right) - I_g\left(\frac{T_r}{\beta}, r-1\right)}, \tag{4.1}$$

where $I_g(x,n) = \int_0^x e^{-t}t^{n-1}$ is the incomplete gamma function.

The Bayes estimator under SELF is given by

$$\hat{\theta}_S = \left[\frac{I_g\left(\frac{T_r}{\alpha}, r-2\right) - I_g\left(\frac{T_r}{\beta}, r-2\right)}{I_g\left(\frac{T_r}{\alpha}, r-1\right) - I_g\left(\frac{T_r}{\beta}, r-1\right)}\right] T_r. \tag{4.2}$$

The Bayes estimator of θ under linex loss function, say $\hat{\theta}_A$, is given by

$$e^a \frac{I_g\left(\frac{T_r}{\alpha}, r\right) - I_g\left(\frac{T_r}{\beta}, r\right)}{I_g\left(\frac{T_r - a\hat{\theta}_A}{\alpha}, r\right) - I_g\left(\frac{T_r - a\hat{\theta}_A}{\beta}, r\right)} = \left(\frac{T_r}{T_r - a\hat{\theta}_A}\right)^r \tag{4.3}$$

In this case risk functions cannot be obtained in a closed form.

5. The Comparison and Recommendation

It is evident from the equations (2.2),(2.3),(3.2),(3.3),(4.2) and (4.3) that Bayes estimators of the shape parameter of the finite range distribution, under squared error, linex loss functions using quasi, natural conjugate and uniform priors, have different expressions for their definitions. The Bayes estimators do depend upon the parameters of the prior distributions.

In figure-l we have plotted the risk functions B_1 and B_2, of the Bayes estimators $\hat{\theta}_S$ and $\hat{\theta}_A$ respectively, under squared error loss function, as given in equation (2.4) and (2.5) for $a = 1$, $r = 5(5)20$ and $d = 0.5(0.5)5.0$.

In figure-2 we have plotted the risk functions C_1 and C_2, of the Bayes estimators $\hat{\theta}_S$ and $\hat{\theta}_A$ respectively, under linex loss function, as given in equation (2.6) and (2.7) for $a = 1$, $r = 5(5)20$ and $d = 0.5(0.5)5.0$.

From figure-l and 2 it is clear that neither of the estimators uniformly dominates the other.

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