South East Asian J. Math. & Math. Sc.
Vol.2 No.1(2003), pp.75-80

PROBLEMS ON FUNCTIONS OF BOUNDED BOUNDARY AND RADIAL ROTATIONS

Susheel Chandra
Department of Applied Sciences
M. M. M. Engineering College, Gorakhpur-273010, India

(Received: August 29, 2003; Submitted by R.C. Srivastava)

Abstract: Coefficient problems related to the functions of bounded boundary and radial rotaions in the unit disc have been obtained.

Keywords and Phrases: Bounded variation, starlike, convex, close-to-convex

1. Introduction

Let $BV[0, 2\pi]$ be the class of all real valued functions and of bounded variations in $[0, 2\pi]$. We denote V_k, the set of functions,

$$f(z) \;=\; z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $D = \{z : |z| < 1\}$ and satisfy

$$f'(z) \;=\; \exp\left\{\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})^{-1} du(t)\right\} \tag{1.2}$$

where $u(t) \in BV[0, 2\pi]$ with

$$\int_0^{2\pi} du(t) \;=\; 2\pi, \quad \int_0^{2\pi} |du(t)| \;\leq\; k\pi \tag{1.3}$$

Similarly, we denote R_k, the class of functions $f(z)$ of the form (1.1) which are regular in D and satisfy

$$f(z) \;=\; z\exp\left\{\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})^{-1} du(t)\right\} \tag{1.4}$$

where $u(t) \in BV[0, 2\pi]$ and satisfies (1.3).

Goodman [1] and Umezava [4] have defined multivalently convex, starlike and close-to-convex functions which are regular in D.

Taking into account the properties of the multivalently convex and starlike functions in D, we discuss some properties of certain classes of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \tag{1.5}$$

which are regular in D (p is a positive integer).

Let $w = f(z)$ of the form (1.5) be a regular function in D with $f'(z) \neq 0$ for $0 < |z| < 1$. Let C_r be the image of $|z| = r$ $(0 < r < 1)$ under the mapping $f(z)$. If C_r has boundary rotation atmost $pk\pi$ then $f(z)$ will be called a function of bounded boundary rotation of type p. It is clear that $k \geq 2$.

Similarly, Let C'_r denotes the image of $|z| = r$ under the mapping $f(z)$ of the form (1.5) which is regular in D. If C'_r has radial rotation atmost $kp\pi$, then $f(z)$ will be called a function of bounded radial rotation of type p.

From above discussions we now define by $V_{k,p}$, $k \geq 2$, the class of all functions $f(z)$ of the form (1.5) which are regular in D with $f'(z) \neq 0$ in $0 < |z| < 1$, and having the boundary rotation atmost $kp\pi$ of type p. A function $f(z)$ given by (1.5) belongs to $V_{k,p}$, if and only if, $f'(z) \neq 0$ for $0 < |z| < 1$, and with $z = re^{i\theta}$,

$$\int_0^{2\pi} \left| Re \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} \right| \, d\theta \leq \; kp\pi. \tag{1.6}$$

Equivalently as done by Paatero [3] for the class V_k, it can be easily shown that $f(z) \in V_{k,p}$, if and only if, there exists a function $u(t) \in BV[0, 2\pi]$ satisfying

$$\int_0^{2\pi} du(t) \; = \; 2\pi p, \quad \int_0^{2\pi} |du(t)| \; \leq \; kp\pi \tag{1.7}$$

and such that

$$f'(z) \; = \; pz^{p-1} \exp \left\{ \frac{1}{\pi} \int_0^{2\pi} \log(1 - ze^{-it})^{-1} du(t) \right\}. \tag{1.8}$$

Similarly for fixed $k \geq 2$, let $R_{k,p}$ denote the class of functions $f(z)$ of the form (1.5) which are regular and $f(z)/z^p \neq 0$ in D with radial rotation atmost $kp\pi$ of type p. A function $f(z)$ given by (1.5) belongs to $R_{k,p}$, if and only if $f(z) \neq 0$ in $0 < |z| < 1$, and with $z = re^{i\theta}$,

$$\int_0^{2\pi} \left| Re \left\{ \frac{zf'(z)}{f(z)} \right\} \right| \, d\theta \leq \; kp\pi. \tag{1.9}$$

It can be easily shown that $f(z) \in R_{k,p}$, if and only if there exists a function $u(t) \in BV[0, 2\pi]$ satisfying the conditions (1.7) and such that

$$f(z) \;=\; z^p \exp\left\{\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})^{-1} du(t)\right\} \tag{1.10}$$

Let $V_{k,p}(q, \alpha)$ and $R_{k,p}(q, \alpha)$ are classes of functions $f(z)$ of the forms

$$f'(z) \;=\; pz^{p-1} \exp\left\{\frac{(p-\alpha)}{pq\pi}\int_0^{2\pi} \log(1 - z^q e^{-iqt})^{-1} du(t)\right\} \tag{1.11}$$

and

$$f(z) \;=\; z^p \exp\left\{\frac{(p-\alpha)}{pq\pi}\int_0^{2\pi} \log(1 - z^q e^{-iqt})^{-1} du(t)\right\} \tag{1.12}$$

respectively, where $0 \le \alpha < p$ and $u(t)$ satisfies the condition (1.7), and both p and q are positive integers.

It is easy to show that $f(z) \in V_{k,p}(q, \alpha)$ if and only if $g(z) \in R_{k,p}(q, \alpha)$ where

$$g(z) \;=\; \frac{zf'(z)}{p} \tag{1.13}$$

In this paper we have solved a coefficient problem for the class $R_{k,p}(q, \alpha)$. On the basis of identification given by (1.13), we can obtain similar result for $V_{k,p}(q, \alpha)$.

2. Main Results

Noonan [2] has solved the following coefficient problem for the class V_k.

Theorem [2]. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V_k$ and $F_k(z) = z + \sum_{n=2}^{\infty} A_n(k) z^n$ be given by

$$F_k(z) \;=\; \frac{1}{k}\left\{\frac{(1+z)^{k/2}}{(1-z)^{k/2}} - 1\right\}.$$

Then for $n \le \left[\frac{k+6}{4}\right]$, we have $|a_n| \le |A_n(k)|$ with equality for any $n \le \left[\frac{k+6}{4}\right]$, if and only if, $f(z) = z^{-i\theta} F_k(ze^{i\theta})$ for some $\theta \in [0, 2\pi]$.

In this paper we extend this result to the class $R_k(q, \alpha)$. To prove our main result we need the following lemmas.

Lemma 2.1. $f(z) \in R_{k,p}(q, \alpha)$, if and only if, there exists $s_1(z)$ and $s_2(z) \in R_{2,1}(q, 0)$ such that

$$f(z) \;=\; z^p \frac{[s_1(z)/z]^{(k+2)(p-\alpha)/4}}{[s_2(z)/z]^{(k-2)(p-\alpha)/4}}.$$

Lemma 2.2. Let $s(z) \in R_{2,1}(q,0)$ and $B > 0$. Take $g(z) = (s(z)/z)^{-B} = 1 + \sum_{n=1}^{\infty} b_{nq} Z^{nq}$. Then for $B-(n-1)q \geq 0$.

$$|b_{nq}| \;\leq\; \frac{2B(2B-q)(2B-2q)\cdots\{2B-(n-1)q\}}{q^n\;(n)!}, \quad n=1,2\cdots \tag{2.1}$$

and also for $B-q \geq 0$,

$$|b_{nq}| \;\leq\; \frac{2B(3q-2B)(6q-2B)\cdots\{3(n-1)q-2B\}}{(n)!\;q^n}, \quad n=2,3\cdots \tag{2.2}$$

and $|b_{1q}| \leq \dfrac{2B}{q}$.

Lemma 2.3. Let $s(z) \in R_{2,1}(q,0)$ and $\alpha > 0$. Take $g(z) = (s(z)/z)^{\alpha} = 1 + \sum_{n=1}^{\infty} b_{nq} z^{nq}$.
Then

$$|b_{nq}| \;\leq\; \frac{2\alpha(2\alpha+q)(2\alpha+2q)\cdots\{2\alpha-(n-1)q\}}{q^n\;(n)!}, \quad n=1,2\cdots$$

Lemma 2.1 follows from (1.12).

Proof of Lemma 2.2. Let P_q be the class of the functions

$$p(z) \;=\; 1+\sum_{n=1}^{\infty} p_{nq} z^{nq},$$

which are regular and $\mathrm{Re}\,p(z) > 0$ in D.

Since $g(z) = (s(z)/z)^{-B}$, we have

$$1-\frac{zg'(z)}{Bg(z)} \;=\; z\frac{s'(z)}{s(z)} \tag{2.3}$$

and $p(z) = 1-\dfrac{zg'(z)}{Bg(z)} \in P_q$ as $s(z) \in R_{2,1}(q,0)$ (a sub class of starlike functions).

Hence $[p(z)]^{-1} = 1+\sum_{m=1}^{\infty} u_{mq} z^{mq}$ also belongs to P_q. From (2.3) we have

$$(p(z))^{-1}[Bq(z)-zg'(z)] \;=\; Bg(z) \tag{2.4}$$

By equating the coefficient z^{nq} in the above relation we have

$$nqb_{nq} \;=\; \sum_{m=0}^{n-1}(B-mq)b_{mq}u_{(n-m)q}. \tag{2.5}$$

Case I. Let $(n-1)q \leq B$. Applying induction argument and using the fact that $|u_{mq}| \leq 2$, we get the required result.

Case II. Let $q \geq B$. It follows that $(n-1)q \geq B$, $n = 2, 3, \cdots$.

We can write (2.5) as follows:

$$nqb_{nq} \;=\; Bu_{nq} - \sum_{m=0}^{n-1}(mq - B)b_{mq}u_{(n-m)q}.$$

Again by using induction argument we get (2.2).

Proof of Lemma 2.3. Follows as above.

Theorem 2.1. Let $f(z) = z^p + \sum_{n=1}^{\infty} a_{nq} + pz^{nq+p}$ and let

$F_k(z) \;=\; z^p + \sum_{n=1}^{\infty} A_{nq+p}z^{nq+p}$ be given by

$$F_k(z) \;=\; \frac{z^p(1+z^q)^{(k-2)(p-\alpha)/2q}}{(1-z^q)^{(k+2)(p-\alpha)/2q}} \tag{2.6}$$

Then for $(n-1)q \leq \dfrac{(k-2)(p-\alpha)}{4}$, we have $|a_{nq+p}| \leq |A_{nq+p}|$ and result is sharp. Again for $q \geq \dfrac{(k-2)(p-\alpha)}{4}$,

$$|a_{nq+p}| \;\leq\; \sum_{j=0}^{n}\Bigg[\frac{(k+2)(p-\alpha)\{(k+2)(p-\alpha)+1.2q\}\{(k+2)(p-\alpha)+2.2q\}\cdots}{2^j\;q^j\;(j)!}$$

$$\frac{\{(k+2)(p-\alpha)+2(j-1)q\}2B'(2.3q-2B')(2.6q-2B')\cdots\{2.3(n-j-1)-2B'\}}{2^{n-j}q^{n-j}(n-j)!}\Bigg] \tag{2.7}$$

where $B' = (k-2)(p-\alpha)/2$.

Proof. Let $f(z) = z^p + \sum_{n=1}^{\infty} a_{nq+p}z^{nq+p} \in R_{k,p}(q,\alpha)$, then from Lemma 2.1 we have

$$f(z) \;=\; \frac{z^p\{S_1(z)/z\}^{(k+2)(p-\alpha)/4}}{\{S_2(z)/z\}^{(k-2)(p-\alpha)/4}},$$

where $S_1(z)$ and $S_2(z) \in R_{2,1}(q,0)$. Let $[S_1(z)/z]^{(k+2)(p-\alpha)/4} = \sum_{j=0}^{\infty} C_{jq}z^{jq}$ and $[S_2(z)/z]^{-(k-2)(p-\alpha)/4} = \sum_{j=0}^{\infty} B_{jq}z^{jq}$.

Therefore, $f(z) = z^p \left(\sum_{j=0}^{\infty} C_{jq} z^{jq}\right) \left(\sum_{s=0}^{\infty} B_{sq} z^{sq}\right)$.

Equating the coefficient of z^{nq+p}, we get

$$a_{nq+p} = \sum_{j=0}^{n} C_{jq} B_{(n-j)q}.$$

Using (2.1) of Lemma 2.2 and Lemma 2.3, we have

$$|a_{nq+p}| \leq \sum_{j=0}^{n} \Bigg[\frac{(k+2)(p-\alpha)\{(k+2)(p-\alpha)+1.2q\}\{(k+2)(p-\alpha)+2.2q\}\cdots}{2^j\ q^j\ (j)!}$$
$$\frac{\{(k+2)(p-\alpha)+2(j-1)q\}(k-2)(p-\alpha)\{(k-2)(p-\alpha)-1.2q\}}{2^{n-j}}$$
$$\frac{\{(k-2)(p-\alpha)-2.2q\}\{(k-2)(p-\alpha)-2(n-j-1)q\}}{q^{n-j}(n-j)!} \Bigg] \tag{2.8}$$

with $(n-1)q \leq \dfrac{(k-2)(p-\alpha)}{4}$. It can be seen that $|A_{nq+p}|$ is equal to the value of R.H.S of (2.8). The sharpness of the result is followed from the fact that $F_k(z) \in R_{k,p}(q,\alpha)$.

Again if we take $q \geq \dfrac{(k-2)(p-\alpha)}{4}$, it follows that $(n-1)q \geq \dfrac{(k-2)(p-\alpha)}{4}$ for $n = 2, 3\cdots$. (2.7) follows as above by using (2.2) of Lemma 2.2. This completes the proof of the Theorem.

With the relationship between $V_{k,p}(q,\alpha)$ and $R_{k,p}(q,\alpha)$ given by (1.13), we can also get coefficient problem for the class $V_{k,p}(q,\alpha)$ similar to those given in Theorem 2.1.

References

[1] Goodman, A. W., *On the Schwarz-Christoffel Transformation and p-valent functions*, Trans. Amer. Math. Soc. **68** (1948), 204-223.

[2] Noonan, J. W., *Coefficient of function with bounded boundary rotaions*, Proc. Amer. Math. Soc. **29** (1971), 307-312.

[3] Paatero, V., *Über die konforme Abbildungen von Gebiesten deren Räden von beschränkster Drehing sind.*, Ann. Acad. Sci. Fenn Ser. A **33**(9) (1931).

[4] Umezava, T., *Multivalently close-to-convex functions*, Proc. Amer. Math. Soc. **8** (1957), 867-874.