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A STUDY OF MATHEMATICAL MODELS OF PRODUCTION FUNCTIONS

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Abstract : In this paper we study the production and cost function in generalized form. The formulation and generalization may be used to study the production behaviour and cost structure of any industry.

1. Introduction

Production is a sequence of technical processes requiring either directly or indirectly the mental and physical skill of craftsman and consists of changing the shape size and properties of materials and ultimately converting them into more useful articles. In short we may understand production as an organised activity of transformation of raw materials into finished products to satisfy the human wants [5].

Production function is a purely technical relation which connects factors input and outputs. It describes the law of productions, that is the transformation of factor inputs into outputs at any particular time period. The production function represents the technology of an industry on of the economy as a whole. When one is concerned with the input output relation of a single industry this comes under study of micro-economic, No-one is really interested in this type of model. A model which deals with input-output relation of whole industry comes under micro-economic level [10].

If we consider the input of an industry only labour and capital, we will not be able to get realistic model for appropriate measurement of output. We have to take into account managerial skills, good labour relation, technical progress and many other parameter which influence the output. Griliches [7] and Jorgenson [9] studied the effects of various inputs on the production function. Mathematical models in industry have been extensively investigated by Hall [8], Anderson [2,3] and Andrews [1].

In the present work we study the production function and cost function in generalized form.

2. Mathematical Model of Production Function

The mathematical model of production function is based on the assumption that a industry manufactures single product.

If Y denote output, L labour potential, K capital, R raw material, S land input, ν return to scale, γ efficiency parameter, the general mathematical form of production function is

$$Y = f(L, K, R, S, \nu, \gamma) \tag{1}$$

all the variables are flow variables that is they are measured per unit time.

In its general form production function is a purely technological relation between quantities of inputs and quantities of output. When we study the variables R and S we see that raw material comes as part of output and land input for one industry is constant hence does not eneter into an aggregate production function. Hence raw material and land input are included in variable K. Hence the production function now takes the form

$$X = f(L, K, \nu, \gamma) \tag{2}$$

where $X = Y - R$.

The factor ν return to scale refers to the long run analysis of the law of production since it assumes the change in plant of the industry. The efficiency parameter γ refers to the organisational aspect of the production. Two industries with identical factors inputs may have different levels of output due to difference in their organisational efficiencies with the development of scientific technique, human resources are changing very fast which adds to the organisational efficiency. In other words γ changes with time.

The marginal product of a factor is defined as the change in output resulting from a very small change in input factor. Keeping all other factors constant. Marginal product of factor labour is written as $(MP)_L$ and so on. Mathematically we write

$$(MP)_L = \frac{\partial X}{\partial L} \text{ and } (MP)_K = \frac{\partial X}{\partial K}$$

when marginal product is positive

$$(MP)_L > 0 \text{ then } \frac{\partial^2 X}{\partial L^2} < 0$$
$$(MP)_K > 0 \text{ then } \frac{\partial^2 X}{\partial K^2} < 0$$

The law of production describes the technically possible ways of increasing the level of production. Output may be increased in various ways. Output may be increased by increasing all factors of production. This is possible only in the long run. Thus the law of return to scale refers to the long run analysis of the production.

3. The Cobb-Douglas Model

The specific form of production functions was invented in 1928 by the two Americans Cobb-Douglas. This form is most popular in applied research [6]

$$X = b_0 L^{b_1} K^{b_2} \tag{3}$$

for two inputs labour L and capital K and b_0, b_1, b_2 are coefficients of inputs.

Due to fast developments of automation and computer added production, labour factor is not specifically defined in many industries and capital input is implicitily involved in all the inputs. Therefore instead of only two input factors L and K we take n inputs factors. This gives more general form of production function.

The generalized form of production function for n inputs $F_1, F_2, \cdots, F_n$ is

$$X = b_0 F_1^{a_1} F_2^{a_2} \cdots F_n^{a_n} \tag{4}$$

Now we calculate the marginal product of factors for general model

[a]
$$\begin{aligned}(MP)_1 &= \frac{\partial X}{\partial F_1} = b_0 a_1 F_1^{a_1-1} F_2^{a_2} \cdots F_n^{a_n} \\ &= \frac{a_1}{F_1} b_0 F_1^{a_1} F_2^{a_2} \cdots F_n^{a_n} \\ &= \frac{a_1}{F_1} X \\ &= a_1 \frac{X}{F_1} \\ &= a_1 (AP)_1\end{aligned}$$

where $(AP)_1 =$ the average product of input F_1. Similarly for any input

$$(MP)_i = a_i (AP)_i \,, \quad i = 1, 2, \cdots, n$$

[b] Marginal rate of substitution for two factors F_1, F_2 is

$$\begin{aligned} (MRS)_{1,2} &= \frac{\partial X/\partial F_1}{\partial X/\partial F_2} \\ &= \frac{a_1 X/F_1}{a_2 X/F_2} \\ &= \frac{a_1}{a_2}\,\frac{F_2}{F_1} \end{aligned}$$

Marginal rate of substitution for F_i, F_{i+1} factor is

$$(MRS)_{i,i+1} = \frac{a_i}{a_{i+1}}\,\frac{F_{i+1}}{F_i}$$

4. Long Run Analysis of Production Function

In the long run expansion of output may achieved by varying all the factors. in this analysis all the factors are variable and output may increased by changing all the factors by the same proportion or by different proportion.

Let the initial level of production function is

$$X_0 = b_0 F_1^{a_1} F_2^{a_2} \cdots F_n^{a_n}$$

we increase all the factors in the same proportions K and we get a new level of output $X^\star$

$$\begin{aligned} X^\star &= b_0 K^{a_1+a_2+\cdots+a_n} . F_1^{a_1} F_2^{a_2} \cdots F_n^{a_n} \\ X^\star &= K^{a_1+a_2+\cdots+a_n} X_0 \\ X^\star &= K^{\nu} X_0 \end{aligned}$$

This shows that production function is homogeneous. The power ν of K is called the degree of homogeneity of the function and is measure of the return to scale:

If $\nu = 1$ we have constant return to scale. This production is some time called linear homogeneous.

If $\nu > 1$ we have increasing return to scale.

If $\nu < 1$ we have decreasing return to scale.

This general model satisfies the properties of specific model of Cobb-Douglas.

5. Equilibrium Condition of Industry

An industrial organisation wants to maximize its output. The total output and prices factors are given.

Maximize the total output X

$$X \;=\; f(F_i) \quad i = 1, 2, \cdots, n$$

subject to total cost outlay of industry

$$C \;=\; \sum_{i=1}^{n} c_i \, F_i$$

where c_i are price factor of its input F_i. We use the method of Lagrangian multiplier [4]

$$\bar{C} - \sum_{i=1}^{n} C_i \, F_i \;=\; 0$$

$\bar{C}$ is specific value of C.

Multiply the constraint by a constant λ which is the Lagrangian multiplier.

$$\lambda(\bar{C} - \sum_{i=1}^{n} C_i \, F_i) \;=\; 0$$

Lagrangian multipliers are undefined constant which are used for solving constraints, maxima and minima.

We form a composit function

$$\phi \;=\; X + \lambda(\bar{C} - \sum_{i=1}^{n} C_i \, F_i)$$

The maximization of function ϕ implies the maximization of output X. For maximization of ϕ,

$$\frac{\partial \phi}{\partial F_i} \;=\; \frac{\partial X}{\partial F_i} + \lambda(-C_i) \;=\; 0$$

$$\frac{\partial \phi}{\partial \lambda} \;=\; \bar{C} - \sum_{i=1}^{n} c_i \, F_i \;=\; 0$$

$$\lambda \;=\; \frac{\partial X}{\partial F_i} / C_i$$

$$\lambda \;=\; \frac{\partial X}{\partial F_1} / C_1 \;=\; \frac{\partial X}{\partial F_2} / C_2$$

$$\text{or} \qquad \frac{\partial X/\partial F_1}{\partial X/\partial F_2} = \frac{C_1}{C_2}$$

$$\text{or} \qquad \frac{(MP)_1}{(MP)_2} = \frac{C_1}{C_2}$$

Thus the industry is in equilibrium i.e. ratio of marginal productivities factors are equal to ratio of their prices.

6. Cost Function For Generalized Model

Our mathematical model of production function is

$$X = b_0 F_1^{a_1} F_2^{a_2} \cdots F_i^{a_i} \cdots F_n^{a_n}$$

and cost equation is

$$C = \sum_{i=1}^{n} c_i \, F_i$$

where C_i is price of input F_i.

We want to find out cost function C

$$C = f(X)$$

We take this problem as constraint output maximization problem.

$$\text{Maximization} \quad X = b_0 F_1^{a_1} F_2^{a_2} \cdots F_n^{a_n}$$
$$\text{subject to} \quad \bar{C} = \sum_{i=1}^{n} C_i F_i$$

We form a composit function

$$\phi = X + \lambda(\bar{C} - \sum_{i=1}^{n} C_i \, F_i)$$

$\bar{C}$ is a given value of c for the industry i.e. industry has a given amount of money to spend on all the factors of production, λ is a Langrangian multiplier

$$\frac{\partial \phi}{\partial F_i} = a_1 \frac{X}{F_i} - \lambda C_i = 0$$

$$\frac{\partial \phi}{\partial \lambda} = \bar{C} - \sum_{i=1}^{n} C_i \, F_i = 0$$

$$\text{or} \qquad a_i \frac{X}{F_i} = \lambda C_i$$

$$\text{or} \qquad a_1 \frac{X}{F_1} = \lambda C_1$$

$$a_2 \frac{X}{F_2} = \lambda C_2$$

$$\text{or} \qquad \frac{a_1}{a_2}.\frac{F_2}{F_1} = \frac{C_1}{C_2}$$

$$\begin{aligned} F_2 &= \frac{C_1}{C_2}.\frac{a_2}{a_1}.F_1 \\ F_3 &= \frac{C_1}{C_3}.\frac{a_3}{a_1}.F_1 \\ &\vdots \\ F_n &= \frac{C_1}{C_n}.\frac{a_n}{a_1}.F_1 \end{aligned}$$

Hence

$$\begin{aligned} X &= b_0 F_1^{a_1} \left(\frac{c_1}{c_2}\ \frac{a_2}{a_1}\ F_1\right)^{a_2} \left(\frac{c_1}{c_3}\ \frac{a_3}{a_1}\ F_1\right)^{a_3} \cdots \left(\frac{c_1}{c_n}\ \frac{a_n}{a_1}\ F_1\right)^{a_n} \\ X &= b_0[\alpha_2\alpha_3\cdots\alpha_n]F_1^{a_1+a_2+\cdots+a_n} \\ X &= b_0[\alpha_2\alpha_3\cdots\alpha_n]F_1^{\nu} \end{aligned}$$

where $\nu = a_1 + a_2 + \cdots + a_n$ and $\alpha_i = \left(\frac{c_1}{c_i}\ \frac{a_i}{a_1}\right)^{a_i}$, $i = 2, 3, \cdots, n$

$$F_1 = \frac{1}{[b_0(\alpha_2, \alpha_3\cdots\alpha_n)]^{1/\nu}}.X^{1/\nu}$$

Cost equation is

$$\begin{aligned} C &= c_1F_1 + c_2F_2 + \cdots + c_nF_n \\ &= C_1F_1 + C_2\frac{c_1}{c_2}\ \frac{a_2}{a_1}F_1 + C_3\frac{c_1}{c_3}\ \frac{a_3}{a_1}F_1 + \cdots + C_2\frac{c_1}{c_n}\ \frac{a_n}{a_1}F_1 \\ &= (a_1 + a_2 + a_3 + \cdots + a_n)\frac{C_1F_1}{a_1} \end{aligned}$$

$$C = \frac{\nu c_1/a_1}{[b_0(\alpha_2\cdots\alpha_n)]^{1/\nu}}\ X^{1/\nu}$$

This is required cost function. This gives the cost expressed in the terms of output X, production coefficients b_0, a_1 and price factor c_1. The sum $\nu = a_1 + a_2 + \cdots a_n$ is a measure of the return to scale for the generalized model.

If prices factors are given cost depends only on output and we get a functional relation $c = f(X)$.

7. Results and Discussion

In the present work we discussed the general form of production function and Cobb-Douglas model. A generalized model is proposed and marginal product, long run analysis equilibrium condition of industry is studied for new model. The cost function of generalized model is derived. The formulation and generalization may be used to study the production and cost structure of any industry.

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