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ON M-PROJECTIVELY RECURRENT SASAKIAN MANIFOLDS

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Abstract: The object of this paper is to study M-projectively recurrent Sasakian manifolds.

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1. Introduction

Let $M^{2m+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with contact from η, the associated vector field ξ, (1,1)-tensor field φ and the associated Riemannian metric g. If ξ is a Killing vector field, then M^{2m+1} is called a K-contact Riemannian manifold [1,3]. A K-contact Riemannian manifold is called Sasakian [1] if and only if

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X \tag{1.1}$$

holds for all vector fields X, Y, where ∇ denotes the operator of covariant differentiation with respect to g. A Sasakian manifold is K-contact but not conversely. However, a 3-dimensional K-contact manifold is Sasakian.

Recently, R. H. Ojha [2] studied M-projectively flat Sasakian manifold and proved that an M-projectively recurrent Sasakian manifold is M-projectively flat if and only if it is an Einstein manifold. In the present paper we study M-projectively recurrent Sasakian manifolds and it is proved that such a manifold is always an Einstein manifold and hence M-projectively flat.

2. Preliminaries

If R, S, r denote respectively the curvature tensor of type (1,3), the Ricci tensor of type (0,2) and the scalar curvature of a Sasakian manifold M^{2m+1}, then the following relations hold [1,3,4]:

$$\text{a)} \quad \varphi\xi = 0, \quad \text{b)} \quad \eta(\xi) = 1, \quad c) \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$\text{a)}\quad \varphi^2 X = -X + \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$\text{a)}\quad (\nabla_X \varphi)(Y) = R(\xi, X)Y, \quad \text{b)}\quad \nabla_X \xi = -\varphi X, \tag{2.3}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.4}$$

$$S(X,\xi) = 2m\eta(X) \tag{2.5}$$

for any vector fields X, Y, Z.

The above results will be used in the next section.

3. M-Projectively Recurrent Sasakian Manifolds

This section deals with a Sasakian manifold which is M-projectively recurrent. A Sasakian manifold is said to be M-projectively recurrent if it satisfies

$$(\nabla_W M)(X,Y)Z = A(W)M(X,Y)Z, \tag{3.1}$$

where A is a non-zero 1-form and the M-projective curvature tensor M in a Sasakian manifold is given by

$$M(X,Y)Z = R(X,Y)Z - \frac{1}{4m}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \tag{3.2}$$

Q is the Ricci-operator i.e., $g(QX,Y) = S(X,Y)$.

We now define a function f on the manifold M^{2m+1} by $f^2 = g(M,M)$, where the Riemannian metric g is extended to the inner product between the tensor fields in the standard fashion. Then since $\nabla M = A \otimes M$, we get

$$\begin{aligned} f(Yf) &= f^2 A(Y). \text{ From this it follows that} \\ Yf &= fA(Y) \quad (\text{because } \ f \neq 0). \end{aligned} \tag{3.3}$$

From (3.3) we obtain

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + (XA(Y))f.$$

Hence

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X)\}f.$$

Therefore we get $(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})f = \{XA(Y) - YA(X) - A([X,Y])\}f$.

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M^{2m+1} by our assumption, we obtain

$$dA(X,Y) = 0. \tag{3.4}$$

This means that the 1-form A is closed. Now from (3.1) we get

$$(\nabla_X \nabla_Y M)(U,V)W \;=\; \{XA(Y) + A(X)A(Y)\}M(U,V)W.$$

Hence from (3.4) we have

$$(R(X,Y)M)(U,V)W \;=\; [2dA(X,Y)]M(U,V)W \;=\; 0.$$

Therefore for a M-projectively recurrent manifolds, we have

$$R(X,Y)M \;=\; 0, \tag{3.5}$$

where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y.

By virtue of (2.1), (2.4) and (2.5), it follows from (3.2) that

$$\begin{aligned}\eta(M(X,Y)Z) = \frac{1}{4m}[&S(X,Z)\eta(Y) - S(Y,Z)\eta(X)\\ &+ 2m\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}].\end{aligned} \tag{3.6}$$

Replacing X by ξ in (3.6) and using (2.5) we get

$$\eta(M(\xi,Y)Z) = \frac{1}{4m}[2mg(Y,Z) - S(Y,Z)]. \tag{3.7}$$

Again from (3.6), it follows that

$$\eta(M(X,Y)\xi) \;=\; 0. \tag{3.8}$$

By definition from (3.5) we have

$$\begin{gathered}R(X,Y)M(U,V)W - M(R(X,Y)U,V)W - M(U,R(X,Y)V)W\\ -M(U,V)R(X,Y)W = 0.\end{gathered}$$

This gives us

$$\begin{aligned}&g(R(\xi,Y)M(U,V)W,\xi) - g(M(R(\xi,Y)U,V)W,\xi)\\ &-g(M(U,R(\xi,Y)V)W,\xi) - g(M(U,V)R(\xi,Y)W,\xi) = 0.\end{aligned} \tag{3.9}$$

In view of (2.3) (a) and (1), we obtain from (3.9)

$$M(U,V,W,Y) - \eta(Y)\eta(M(U,V)W) + \eta(V)\eta(M(U,V)W)$$

$$\eta(U)\eta(M(Y,V)W) + \eta(W)\eta(M(U,V)Y) - g(Y,U)\eta(M(\xi,V)W) - g(Y,V)\eta(MU,\xi)W = 0, \tag{3.10}$$

where (3.8) have been used, and $M(U,V,W,Y) = g(M(U,V)W,Y)$.

Let $\{e_i\},\ i = 1.2,\cdots,2m+1$ be an orthogonal basis of the tangent space at any point of the manifold. Then putting $Y = U = e_i$ in (3.10) and taking summation for $1 \leq i \leq 2m+1$, we get by virtue of (3.2), (3.6) and (3.7) that

$$S(V,W) = \alpha g(V,W) + \beta\eta(V)\eta(W) \text{ for all } V,W, \tag{3.11}$$

where

$$\alpha = \frac{4m^2+r}{4m+1} \text{ and } \beta = \frac{4m^2+2m-r}{4m+1}.$$

It follows from (3.11) that

$$r = 2m(2m+1). \tag{3.12}$$

Using (3.12) in (3.11), we have

$$S(V,W) = 2mg(V,W) \text{ for all } V,W. \tag{3.13}$$

From (3.13) we can state the following:

Theorem 1. An M-projectively recurrent Sasakian manifold $M^{2m+1}(\varphi,\xi,\eta,g)$ $(m>1)$ is an Einstein manifold.

In view of (3.13), we obtain from (3.6) that

$$\eta(M(X,Y)Z) = 0 \text{ for all } X,Y,Z, \tag{3.14}$$

which implies that

$$\eta(M(\xi,Y)Z) = 0. \tag{3.15}$$

By virtue of (3.14) and (3.15), we get from (3.10) that

$$M(U,V,W,Y) = 0 \text{ for all } U,V,W,Y.$$

Hence we can state the follows:

Theorem 2. An M-projectively recurrent Sasakian manifold $M^{2m+1}(\varphi,\xi,\eta,g)$ $(m>1)$ is M-projectively flat.

For an M-projectively symmetric Riemannian manifold, we have $\nabla M = 0$ and hence the relation $R(X,Y)M = 0$ holds. Thus we have the following:

Corollary. An M-projectively symmetric Sasakian manifold $M^{2m+1}(\varphi,\xi,\eta,g)$ $(m > 1)$ is M-projectively flat.

References

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