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ON STRONGLY MULTIPLICATIVE GRAPHS

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Abstract : In this note we show that every wheel is strongly multiplicative. Also we give a formula for $\lambda(n)$, the maximum number of edges in a strongly multiplicative graph of order n.

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1. Introduction

In an interesting paper [2] L. W. Beineke and S. M. Hegde have studied strongly multiplicative graphs. In fact they have shown that all graphs like trees, wheels and grids are strongly multiplicative. They have also obtained an upper bound for the maximum number of edges $\lambda(n)$ for a given strongly multiplicative graph of order n. Recently in [1], C. Adiga, H. N. Ramaswamy and D. D. Somashekara have obtained a sharper upper bound for $\lambda(n)$.

In this note we give an alternate proof using Bertrand's hypothesis, of the result that every wheel is strongly multiplicative. Erdös [3] has obtained an asymptotic formula for $\lambda(n)$. In his lecture at the International Conference on Discrete Mathematics and Number Theory, held at Tiruchirapalli, Tamil Nadu, India, L. W. Beineke had posed the problem of obtaining exact formula for $\lambda(n)$. In this note we obtain a formula for $\lambda(n)$ in terms of divisor functions.

We recall the definition of strongly multiplicative graphs given in [2].

Definition. A graph with n vertices is said to be strongly multiplicative if its vertices can be labelled $1, 2, \cdots, n$ so that the values on the edges, obtained as the product of labels of their end vertices, are all distinct.

We give a simple proof of the result in [2] that every wheel is strongly multiplicative. A wheel $W_{n+1} (n \geq 3)$ is a graph with $n+1$ vertices: $v_1, v_2, \cdots v_n, w$, such that $[v_1, v_2, \cdots, v_n, v_1]$ is a cycle $[v_i, w]$ is an edge for $1 \leq i \leq n$.

2. Main Results

Theorem 1. Every wheel is strongly multiplicative.

Proof. Case(i): $n+1$ is odd. Let p be a prime such that $\frac{n}{2} < p < n$. Such a prime p exists by Bertrand's hypothesis [4]. Consider the labelling of the graph:

$$v_1 = 1,\ v_2 = 2, \cdots,\ v_{p-1} = p-1,\ v_p = p+1, \cdots,\ v_n = n+1,\ w = p.$$

Since $n+1 \geq 5$, p is odd and so the number $1\ 2, 2\ 3, \cdots, (p-2)(p-1), (p-1)(p+1), \cdots, n(n+1)$, are all even and strictly increasing. Since $n+1$ is odd, it follows that all the rim edges have different values. The values of the spoke edges are $1p, 2p, \cdots, (p-1)p, (p+1)p, \cdots, (n+1)p$, which are all distinct. Since $\frac{n}{2} < p < n$, no rim value is divisible by p. However, each value on the spoke edge is divisible by p. Hence, the above values of the edges are distinct for the wheel. Hence when $n+1$ is odd the wheel W_{n+1} is strongly multiplicative.

Case (ii): $n+1$ is even. By Bertrand's hypothesis [4] there exists a prime p such that $\frac{n+1}{2} < p < n+1$. Let $m(\neq p)$ be the largest odd integer less than $n+1$. When $m > p$, label each vertex of the wheel as follows:

$$v_i = \begin{cases} i, & 1 \leq i \leq p-1 \\ i+1, & p \leq i < n,\ i \neq (m-1) \\ n+1, & i = m-1 \\ m, & i = n \end{cases}$$

and $w = p$.

When $m < p$, label each vertex of the wheel as follows:

$$v_i = \begin{cases} i, & 1 \leq i \leq m-1 \\ n+1, & i = m \\ i, & m+1 \leq i \leq p-1 \\ i+1, & p \leq i < n \\ m, & i = n \end{cases}$$

and $w = p$.

Since every value of the spoke edge is divisible by p and no value of the rim edge is divisible by p, by the choice of p it follows that values of the edges of the wheel are all distinct. Hence in this case also the graph is strongly multiplicative.

Let $\lambda(n)$ denote the maximum number of edges in a strongly multiplicative graph of order n. Thus

$$\lambda(n) = |\{rs \mid 1 \leq r < s \leq n\}|.$$

Also, let

$$\delta(n) \ = \ \lambda(n) - \lambda(n-1).$$

The number theoretic functions λ and δ were studied in [2]. Now, we give a formula for $\lambda(n)$.

If $k = rs$ where $1 \leq r < s \leq n$, then $1 \leq k \leq n(n-1)$. Consider the set $S = \{1, 2, 3, \cdots, n(n-1)\}$. Note that if $1 \leq k \leq n$, then the number of representations of k in the form $k = rs$ where $1 \leq r < s \leq n$ is $\left[\frac{d(k)}{2}\right]$, where $d(k)$ denotes the number of distinct divisors of k and $[x]$ denotes the largest integer less than or equal to x. If $n < k \leq n(n-1)$, the number of representations of k in the form $k = r - s$ where $1 \leq r \leq s \leq n$ is $\left[\frac{d(k)}{2}\right] - d_n(k)$ where $d_n(k)$ denotes the number of divisors of k greater than n. Thus if $f(k)$ denotes number of representations of k in the form $k = rs$ where $1 \leq r < s \leq n$, then we have

$$f(k) = \begin{cases} \left[\frac{d(k)}{2}\right], & \text{if } 1 \leq k \leq n \\ \\ \left[\frac{d(k)}{2}\right] - d_n(k), & \text{if } n < k \leq n(n-1). \end{cases}$$

Let $g(k) = \min\{1, f(k)\}$. Then $g(k) = 1$ when there is at least one representation of k as $k = rs$ where $1 \leq r < s \leq n$ and $g(k) = 0$ when there is no such representation. Since $\lambda(n)$ is the number of distinct integers of the form $k = rs$ with $1 \leq r < s \leq n$, we have proved:

Theorem 2. $\lambda(n) = \sum\limits_{k=1}^{n(n-1)} g(k).$

References

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