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GROWTH PROPERTIES OF A CLASS OF ENTIRE DOUBLE DIRCHLET SEQUENCES

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Abstract: In this paper, we have defined the growth (order and type) of complex double sequences analogous to the respective definitions in the theory of integral functions represented by double Dirichlet series with fixed sequences $\{\lambda_m\}$ and $\{\mu_n\}$ of exponents. We have considered a class P of entire double Dirchlet sequences (EDDS) and have examined their growth properties. Distribution of elements of P over the universal set U of all EDDS has been investigated and has been depicted by Venn diagram.

1. Introduction

Let $\{\lambda_m\}$ and $\{\mu_n\}$ be strictly increasing and divergent sequence of positive reals satisfying

$$\lim_{m\to\infty} \frac{\log m}{\lambda_m} \;=\; 0 \;=\; \lim_{n\to\infty} \frac{\log n}{\mu_n} \tag{1.1}$$

A double Dirichlet series $f(s_1, s_2) = \sum a_{mn}\, e^{s_1\lambda_m + s_2\mu_n}$ represents an entire function if

$$\lim_{m+n\to\infty} \frac{\log |a_{mn}|}{\lambda_m + \mu_n} \;=\; -\infty \tag{1.2}$$

Throughout this paper, any complex sequence satisfying (1.2) will be called an entire double Dirichlet sequence (EDDS, in short).

The Ritt order, or simply order, $\rho(f)$ of the entire Dirichlet series $f(s) = \sum\, a_n e^{s\lambda_n}$ is defined as

$$\rho(f) \;=\; \lim_{\sigma\to\infty} \sup\, \frac{\log\log M(\sigma, f)}{\log \sigma}$$

where

$$M(\sigma, f) \;=\; \overset{1.u.b.}{-\infty < t < \infty} \,|f(\sigma + it)|$$

A necessary and sufficient condition that $f(s)$ be of finite Ritt order $\rho(f)$ is that

$$\rho(f) \;=\; \lim_{n\to\infty} \sup \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}$$

The type $\tau(f)$ of an entire Dirichlet series $f(s)$ of finite order ρ is defined as

$$\tau(f) \;=\; \lim_{\sigma\to\infty} \sup \frac{\log M(\sigma, f)}{e^{\sigma\rho}}$$

A necessary and sufficient condition for this is

$$\tau(f) \;=\; \lim_{n\to\infty} \sup \frac{\lambda_n}{e^{\rho}} \, |a_n|^{\frac{\rho}{\lambda_n}}$$

Analogous to these characterizations of order and type of entire functions represented by Dirichlet series, we define order and type of entire double Dirichlet sequences as follows:

A double sequence $f = \{f_{mn}\}$ will be said to of order ρ if

$$\rho \;=\; \lim_{m+n\to\infty} \sup \frac{(\lambda_m + \mu_n)\log(\lambda_m + \mu_n)}{\log |f_{mn}|^{-1}} \tag{1.3}$$

A double sequence $f = \{f_{mn}\}$ of order ρ $(0 < \rho < \infty)$ will be said to be of type τ if

$$\tau \;=\; \frac{1}{e\rho} \lim_{m+n\to\infty} \sup \left[(\lambda_m + \mu_n)\, |f_{mn}|^{\frac{\rho}{(\lambda_m+\mu_n)}}\right]. \tag{1.4}$$

Let P be a class of EDDS defined as

$$P \;=\; \left\{ f \;=\; \{f_{mn}\}; (\lambda_m + \mu_n)\, |f_{mn}|^{\frac{1}{(\lambda_m+\mu_n)}} \text{ is bounded} \right\}$$

Obviously, P is a subset of the universal set of U of all EDDS. Let us denote by R_0, R_1 and R the subclasses of U consisting of EDDS with order $\rho < 1$, $\rho = 1$ and $\rho \leq 1$, respectively. Similarly, we denote by T_0, T_1 and T the subclasses of U consisting of all EDDS with type $\tau < 1$ $\tau = 1$ and $\tau \leq 1$, respectively. A^c will denote the complement of A in U, i.e.,

$$A^c \;=\; U - A$$

Clearly,

$$R_0 \cup R_1 \;=\; R; T_0 \cup T_1 \;=\; T;$$

$$R_1 \cap R_0 \;=\; \phi,\; T_0 \cap T_1 \;=\; \phi;$$

2. Growth Properties of Elements of *P*

We first note that if f is any arbitrary element of P then by equation (1.3)

$$\begin{aligned}\rho(f) \;&=\; \lim_{m+n\to\infty}\sup \frac{(\lambda_m+\mu_n)\;\log(\lambda_m+\mu_n)}{\log|a_{mn}|^{-1}} \\ &\leq \lim_{m+n\to\infty}\sup \frac{(\lambda_m+\mu_n)\;\log(\lambda_m+\mu_n)}{\log\left\{\left(\frac{\lambda_m+\mu_n}{K}\right)^{(\lambda_m+\mu_n)}\right\}} = 1.\end{aligned}$$

This implies that order of every element of P is less than or equal to 1. let us denote by P_0 and P_1 the subsets of P containing elements of order < 1 and $=1$, respectively. Thus,

$$R_0 \cap P \;=\; P_0 \;\text{ and }\; R_1 \cap P \;=\; P_1.$$

Theorem 2.1. $R_0 \subset P \subset R$; the set inclusions being proper.

Proof. Let $g = \{g_{mn}\}$ be an arbitrary element of R_0. Then

$$\rho(g) \;=\; \lim_{m+n\to\infty}\sup \frac{(\lambda_m+\mu_n)\;\log(\lambda_m+\mu_n)}{\log|g_{mn}|^{-1}} = r < 1$$

There exist N such that

$$\frac{(\lambda_m+\mu_n)\;\log(\lambda_m+\mu_n)}{\log|g_{mn}|^{-1}} < r \quad \text{whenever } m+n \geq N$$

Thus,

$$\begin{aligned}|g_{mn}| \;&<\; (\lambda_m+\mu_n)^{-(\lambda_m+\mu_n)} \quad [r<1] \\ (\lambda_m+\mu_n)&|g_{mn}|^{\frac{1}{\lambda_m+\mu_n}} \quad r < 1\end{aligned}$$

Hence $g \in P$.

$$\text{Thus }\; R_0 \subseteq P. \tag{2.1}$$

By definition, it is clear that

$$P \subseteq R. \tag{2.2}$$

From equation (2.1) and (2.2), we have

$$R_0 \subseteq P \subseteq R.$$

That the set inclusions are proper will be evident by the Examples 3.3 and 4.4 given in sections 3 and 4, respectively.

Hence the theorem.

Corollary 2.1. $R_0 = P_0$

3. Certain Examples in P

Before proving further results about the growth properties of elements of P, let us first enlist a few important elements of P, which will be referred to quite frequently. In this section we define various elements of P and investigate their growth.

Example 3.1. Consider the elements $a = \{a_{mn}\}$ of P such that

$$a_{mn} \;=\; \frac{\left(1+\frac{1}{m+n}\right)^{(m+n)(\lambda_m+\mu_n)/t}}{(\lambda_m+\mu_n)^{(\lambda_m+\mu_n)/t}}\;;\quad 0<t<1.$$

Obviously $a \in P,\ \rho(a) = t < 1,\ \tau(a) = \frac{1}{t} > 1;\ \forall\ 0 < t < 1.$ In other words $a \in P_0 \cap T^c$.

Example 3.2. Consider the elements $b = \{b_{mn}\}$ such that

$$b_{mn} \;=\; \left(1+\frac{1}{\lambda_m+\mu_n}\right)^{\lambda_m+\mu_n}.$$

Note that $b \in P$, since $\forall\ m, n$

$$(\lambda_m+\mu_n)|b_{mn}|^{\frac{1}{\lambda_m+\mu_n}} = 1.$$

Now order of b is

$$\rho(b) \;=\; \lim_{m+n\to\infty}\ \sup\ \frac{(\lambda_m+\mu_n)\log(\lambda_m+\mu_n)}{\log|b_{mn}|^{-1}} = 1$$

and type of b is

$$\tau(b) \;=\; \frac{1}{e\rho}\lim_{m+n\to\infty}\ \sup\left[(\lambda_m+\mu_n)|b_{mn}|^{\frac{\rho}{\lambda_m+\mu_n}}\right] = \frac{1}{e} < 1.$$

Thus $b \in P$ and $\rho(b) = 1$, $\tau(b) = \dfrac{1}{e} < 1$ in other words $b \in P_1 \cap T_0$.

Example 3.3. Consider the element $c = \{c_{mn}\}$ such that

$$c_{mn} \;=\; \left(\frac{e^2}{\lambda_m + \mu_n}\right)^{\lambda_m+\mu_n}.$$

Note that $c \in P$, since

$$(\lambda_m + \mu_n)|c_{mn}|^{\frac{1}{(\lambda_m+\mu_n)}} = e^2.$$

Also

$$\rho(c) \;=\; \lim_{m+n\to\infty} \sup\, \frac{(\lambda_m + \mu_n)\log(\lambda_m + \mu_n)}{\log|c_{mn}|^{-1}} = 1$$

and

$$\tau(c) \;=\; \frac{1}{e\rho} \lim_{m+n\to\infty} \sup\left[(\lambda_m + \mu_n)|c_{mn}|^{\frac{\rho}{\lambda_m+\mu_n}}\right] = e > 1.$$

Thus $c \in P$ and $\rho(c) = 1$, $\tau(c) = e > 1$. In other words $c \in P_1 \cap T^c$.

Example 3.4. Consider the element $d = \{d_{mn}\}$ such that

$$d_{mn} \;=\; \left(\frac{1}{\lambda_m + \mu_n}\right)^{e(\lambda_m+\mu_n)}.$$

Note that $d \in P$ since

$$(\lambda_m + \mu_n)|d_{mn}|^{\frac{1}{\lambda_m+\mu_n}} = (\lambda_m + \mu_n)^{1-e} \to 0 \text{ as } m + n \to \infty.$$

order of d is

$$\rho(d) \;=\; \lim_{m+n\to\infty} \sup\, \frac{(\lambda_m + \mu_n)\log(\lambda_m + \mu_n)}{e(\lambda_m + \mu_n)\log(\lambda_m + \mu_n)} = \frac{1}{e} < 1$$

and type of d is

$$\tau(d) \;=\; \lim_{m+n\to\infty} \sup\left[(\lambda_m + \mu_n)\;\left(\frac{1}{\lambda_m + \mu_n}\right)\right] = 1.$$

Thus $d \in P$ and $\rho(d) = \dfrac{1}{e} < 1$, $\tau(d) = 1$. In other words $d \in P_0 \cap T_1$.

Example 3.5. Consider the element $i = \{i_{mn}\}$ of P such that

$$i_{mn} \;=\; \left(\frac{e}{\lambda_m + \mu_n}\right)^{\lambda_m+\mu_n}.$$

Firstly we show that $i \in P$. For this, we note that

$$(\lambda_m + \mu_n)|i_{mn}|^{\frac{1}{\lambda_m+\mu_n}} \;=\; e.$$

Now order of i is

$$\rho(i) \;=\; \lim_{m+n\to\infty} \sup \frac{(\lambda_m + \mu_n)\log(\lambda_m + \mu_n)}{(\lambda_m + \mu_n)\log\left(\frac{\lambda_m+\mu_n}{e}\right)} = 1$$

Further

$$\tau(i) \;=\; \frac{1}{e} \lim_{m+n\to\infty} \sup \left[(\lambda_m + \mu_n)\; \left|\left(\frac{e}{\lambda_m + \mu_n}\right)\right|\right] = 1.$$

Thus $i \in P$ and $\rho(i) = 1$, $\tau(i) = 1$. In other words $i \in P_1 \cap T_1$.

Example 3.6. Consider the element $j_t = \{j_{mn}^{(t)}\}$
where

$$j_{mn}^{(t)} \;=\; (\lambda_m + \mu_n)^{-(\lambda_m+\mu_n)/t};\quad 0 < t < 1$$

Firstly we shall show that $j_t \in P$. For this, we note that

$$(\lambda_m + \mu_n)|j_{mn}^{(t)}|^{\frac{1}{\lambda_m+\mu_n}} \;=\; (\lambda_m + \mu_n)^{1-\frac{1}{t}} \quad \to 0 \text{ as } m + n \to \infty \quad [0 < t < 1]$$

Now order of j_t is

$$\rho(j_t) \;=\; \lim_{m+n\to\infty} \sup \frac{(\lambda_m + \mu_n)\log(\lambda_m + \mu_n)}{\frac{(\lambda_m+\mu_n)}{t}\log(\lambda_m + \mu_n)} = t < 1$$

and type

$$\tau(j_t) \;=\; \frac{1}{et} \lim_{m+n\to\infty} \sup \left[(\lambda_m + \mu_n)\; \left|(\lambda_m + \mu_n)^{-1}\right|\right] = \frac{1}{et}.$$

Case I. If $0 < t < \frac{1}{e}$

In this case, order of j_t is

$$\rho(j_t) \;=\; t < \frac{1}{e} < 1$$

and type of j_t is

$$\tau(j_t) \;=\; \frac{1}{et} > 1$$

Therefore, in this case, $j_t \in P_0 \cap T^c$.

Case II. If $t = \dfrac{1}{e}$

In this case, order of j_t is

$$\rho(j_t) \;=\; t = \frac{1}{e} < 1$$

and type of j_t is

$$\tau(j_t) \;=\; \frac{1}{et} = 1$$

Therefore, in this case, $j_t \in P_0 \cap T_1$.

Case III. If $\dfrac{1}{e} < t < 1$

In this case,

$$\rho(j_t) \;=\; t < 1$$

and

$$\tau(j_t) \;=\; \frac{1}{et} < 1$$

Therefore, in this case, $j_t \in P_0 \cap T_0$.

4. Certain Examples in P^c

In this section, we define certain elements of U, which are not members of P. These elements form certain counter examples, which help us in studying the distribution of elements of U into various subclasses of U defined earlier.

Example 4.1. Consider the element $u = \{u_{mn}\}$ such that

$$u_{mn} \;=\; \left(\frac{e^2}{\lambda_m + \mu_n}\right)^{\frac{\lambda_m+\mu_n}{e}}$$

Note that $u \in U$, since

$$\lim_{m+n\to\infty} \frac{\log|u_{mn}|}{\lambda_m+\mu_n} = \lim_{m+n\to\infty} \frac{1}{e}[2\log e - \log(\lambda_m+\mu_n)]$$

$$= -\infty \quad [\text{as} \;\; m+n \to \infty,\; \lambda_m+\mu_n \to \infty]$$

Now order of u is

$$\rho(u) \;=\; \lim_{m+n\to\infty} \sup \frac{e}{1-\frac{2\log e}{\log(\lambda_m+\mu_n)}} \;=\; e > 1$$

This also show that $u \notin P$.

also type of u is

$$\tau(u) \;=\; \frac{1}{e^2} \lim_{m+n\to\infty} \sup \left[(\lambda_m+\mu_n)\frac{e^2}{\lambda_m+\mu_n}\right] = 1.$$

Thus $u \notin P$ and $\rho(u) = e > 1$, $\tau(u) = 1$. In other words $u \in R^c \cap T_1$.

Example 4.2. Consider the element $v = \{v_{mn}\}$, such that

$$v_{mn} \;=\; \left(\frac{e}{\lambda_m+\mu_n}\right)^{K'(\lambda_m+\mu_n)} ;\quad K' < 1$$

Also $v \in U$, since

$$\lim_{m+n\to\infty} \frac{\log|v_{mn}|}{\lambda_m+\mu_n} = \lim_{m+n\to\infty} K'[\log e - \log(\lambda_m+\mu_n)]$$

$$= -\infty \quad [\text{as} \;\; m+n \to \infty,\; \lambda_m+\mu_n \to \infty]$$

The order of v is

$$\rho(v) \;=\; \frac{1}{K'} \lim_{m+n\to\infty} \sup \frac{\log(\lambda_m+\mu_n)}{\log(\lambda_m+\mu_n)\left[1-\frac{\log e}{\log(\lambda_m+\mu_n)}\right]} \;=\; \frac{1}{K'} > 1 \;\; [K' < 1]$$

Hence $v \notin P$.

Type of v is

$$\tau(v) \;=\; \frac{K'}{e} \lim_{m+n\to\infty} \sup \left[(\lambda_m+\mu_n)\left(\frac{e}{\lambda_m+\mu_n}\right)\right] = K' < 1$$

Thus $v \notin P$ and $\rho(v) = \dfrac{1}{K'} > 1$, $\tau(v) = K' < 1$. In other words, $v \in R^c \cap T_0$.

Example 4.3. Consider the element $w = \{w_{mn}\}$ such that

$$w_{mn} \ = \ \left(\frac{e^3}{\lambda_m + \mu_n}\right)^{\frac{(\lambda_m+\mu_n)}{e}}$$

Note that, $w \in U$, since

$$\lim_{m+n\to\infty} \frac{\log|w_{mn}|}{\lambda_m + \mu_n} = \lim_{m+n\to\infty} \frac{1}{e}[3\log e - \log(\lambda_m + \mu_n)]$$
$$= -\infty \quad [\text{as} \quad m+n \to \infty, \ \lambda_m + \mu_n \to \infty]$$

Now order of w is

$$\rho(w) \ = \ \lim_{m+n\to\infty} \sup \frac{e}{1 - \frac{3\log e}{\log(\lambda_m+\mu_n)}} \ = \ e > 1$$

Which also show that $w \notin P$.

Type of w is

$$\tau(w) \ = \ \frac{1}{e^2} \lim_{m+n\to\infty} \sup \left[(\lambda_m + \mu_n)\frac{e^3}{\lambda_m + \mu_n}\right] = e > 1.$$

Thus $w \notin P$ and $\rho(w) = e > 1$, $\tau(w) = e > 1$. In other words $w \in R^c \cap T^c$.

Example 4.4. Consider the element $x = \{x_{mn}\}$ such that

$$x_{mn} \ = \ \left[\frac{\log(\lambda_m + \mu_n)}{\lambda_m + \mu_n}\right]^{\lambda_m+\mu_n}$$

Firstly we shall show that $x \notin P$. For this, we note that

$$(\lambda_m + \mu_n)|x_{mn}|^{\frac{1}{\lambda_m+\mu_n}} \ = \ \log(\lambda_m + \mu_n) \ \text{ which is unbounded}$$

Also that, $x \in U$, since

$$\lim_{m+n\to\infty} \frac{\log|x_{mn}|}{\lambda_m + \mu_n} = \lim_{m+n\to\infty} \log\left[\frac{\log(\lambda_m + \mu_n)}{\lambda_m + \mu_n}\right]$$
$$= -\infty \quad [\text{as} \quad m+n \to \infty, \ \lambda_m + \mu_n \to \infty]$$

Now order of x is

$$\rho(x) \;=\; \lim_{m+n\to\infty} \sup \frac{1}{1-\frac{\log\log(\lambda_m+\mu_n)}{\log(\lambda_m+\mu_n)}} \;=\; 1$$

and type of x is

$$\tau(x) \;=\; \frac{1}{e} \lim_{m+n\to\infty} \sup \log(\lambda_m+\mu_n) > 1.$$

Thus $x \notin P$ and $\rho(x) = 1$, $\tau(x) > 1$. In other words $x \in R_1 \cap P^c \cap T^c$.

5. Distribution Of P_0 and P_1 Over U

In this section, we investigate the distribution of P_0 and P_1 over various subsets of U viz. R_0, R_1, R and T_0, T_1, T defined in section 2. We shall make full use of examples of elements in P and P^c established in section 3 and 4 respectively for this purpose.

Theorem 5.1.

(i) P_1 is not a subset of T.

(ii) $R_1 \cap T_0$ is a proper subset of P, hence of P_1.

Proof.

(i) The sequence 'c' of Example 3.3 is an element of P whose order is 1 and type is greater than 1. Thus 'c' is an element of P_1 but not of T.

(ii) Let f be an arbitrary element of $R_1 \cap T_0$

$$\Rightarrow \rho(f) \;=\; 1 \;\text{ and }\; \tau(f) < 1$$

Therefore, for some r, $0 < r < 1$, we can find N such that,

$$\frac{1}{e\rho}\left[(\lambda_m+\mu_n)\,|a_{mn}|^{\frac{\rho}{\lambda_m+\mu_n}}\right] < r; \;\; \forall\, m+n \geq N$$

$$\Rightarrow (\lambda_m+\mu_n)\,|a_{mn}|^{\frac{1}{\lambda_m+\mu_n}} < r.e; \;\; \forall\, m+n \geq N$$

where $0 < r < 1$, $2 < e < 3$

$$\Rightarrow f \in P.$$

Further, since $R_1 \cap P = P_1$, $R_1 \cap T_0$ is also a suubset of P_1.

To complete the proof, note that the sequence

$$i = \{i_{mn}\}$$

of Example 3.5 is a member of P, which does not belong to T_0 [because $\tau(i) < 1$] and hence $i \notin R_1 \cap T_0$.

Thus $R_1 \cap T_0$ is a proper subset of P, hence of P_1.

In order to give a complete Venn-diagram about various subsets of U, it remains to be checked whether elements of U with order one and type one belong necessarily to P or not.

Theorem 5.2.

$R_1 \cap T_1$ is a proper subset of P hence of P_1.

Proof.

Let $k = \{k_{mn}\}$ be an element of U, whose order $\rho(k) = 1$ and type $\tau(k) = 1$.

we know that

$$\tau(k) \;=\; \frac{1}{e\rho} \lim_{m+n\to\infty} \sup \left[(\lambda_m + \mu_n)\, |k_{mn}|^{\frac{\rho}{\lambda_m+\mu_n}}\right]$$

$$1 \;=\; \frac{1}{e} \lim_{m+n\to\infty} \sup \left[(\lambda_m + \mu_n)\, |k_{mn}|^{\frac{1}{\lambda_m+\mu_n}}\right]$$

or,

$$\lim_{m+n\to\infty} \sup(\lambda_m + \mu_n)\, |k_{mn}|^{\frac{1}{\lambda_m+\mu_n}} \;=\; e$$

This implies that $k = \{k_{mn}\} \in P$.

Hence $R_1 \cap T_1$ is a subset of P and hence of P_1. Further note that the sequence c of Example 3.3 is an element of P whose order 1 and type is $e > 1$. Thus $c \in P_1$ but $c \notin R_1 \cap T_1$.

Hence $R_1 \cap T_1$ is a proper subset of P and hence of P_1.

With the help of examples of section 3 and counter examples of section 4, the distribution of different entire double Dirichlet sequences in various classes of

U is tabulated in the following Table:

TABLE

Sequence	**Example Number**	**Whether belongs to P or Not**	**Order (ρ) of sequence**	**Type (τ) of sequence**	**Position of sequence in Ven diagram**
a	3.1	Yes	$\rho < 1$	$\tau > 1$	$P_0 \cap T^c$
b	3.2	Yes	$\rho = 1$	$\tau = \frac{1}{e} < 1$	$P_1 \cap T_0$
c	3.3	Yes	$\rho = 1$	$\tau = e > 1$	$P_1 \cap T^c$
d	3.4	Yes	$\rho = \frac{1}{e} < 1$	$\tau = 1$	$P_0 \cap T_1$
i	3.5	Yes	$\rho = 1$	$\tau = 1$	$P_1 \cap T_1$
$j_t;\ 0 < t < \frac{1}{e}$	3.6 (case I)	Yes	$\rho = t < \frac{1}{e} < 1$	$\tau = \frac{1}{et} < 1$	$P_0 \cap T^c$
$j_t;\ t = \frac{1}{e}$	3.6 (case II)	Yes	$\rho = t = \frac{1}{e} < 1$	$\tau = \frac{1}{et} = 1$	$P_0 \cap T_1$
$j_t;\ \frac{1}{e} < t < 1$	3.6 (case III)	Yes	$\rho = t < 1$	$\tau = \frac{1}{et} < 1$	$P_0 \cap T_0$
u	4.1	No	$\rho = e > 1$	$\tau = 1$	$R^c \cap T_1$
v	4.2	No	$\rho > 1$	$\tau < 1$	$R^c \cap T_0$
w	4.3	No	$\rho = e > 1$	$\tau = e > 1$	$R^c \cap T^c$
x	4.4	No	$\rho = 1$	$\tau > 1$	$R_1 \cap P^c \cap T^c$

All the above results can also be complied into the Venn diagram page 42, common boundaries in the diagram should essentially be considered as empty sets. All the sets are given rectangular shapes, labelled at the corners of the principal diagonal.

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