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A FIXED POINT THEOREM FOR THREE MAPPINGS

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Abstract: In this paper two fixed point theorems for three mappings have been proved.

Keywords and Phrases: Self maps, common fixed point, complete metric space.

1. Introduction

A well known Banach contraction principle states that a contraction mappings on a complete metric space has a unique fixed point. Jaggi and Das [1] in 1980 gave an extension of Banach fixed point theorem through a rational expressions. This result was generalized by Murthy and Sharma [2] in 1991. In this paper, we prove two fixed point theorems for three self mappings.

2. Main Results

We establish the following theorems:

Theorem 1. Let E, F and T be the three self maps of a complete metric space (X, d) satisfying the following conditions:

(a) (E, T) and (F, T) are commuting pairs.

(b) $EX \subset TX$, $FX \subset TX$.

(c) There exist integers $r, s > 0$ such that

$$d(E^r x, F^s y) \;\leq\; \frac{K.d(Tx, E^r x).d(Ty, F^s y)}{d(Ty, F^s y) + d(Ty, E^r x)}$$

for every $x, y \in X$ and $0 < K < 1$.

Then E, F and T have a unique common fixed point in X, provided T is continuous.

Proof. Using (a) and (b)

$$E^rT = TE^r, \quad F^sT = TF^s \tag{1}$$

and

$$E^rX \subset EX \subset TX, \quad F^sX \subset FX \subset TX \tag{2}$$

Let x_0 be any arbitrary point in X. Since $E^rX \subset TX$. We can choose a point x_1 in X such that $Tx_1 = E^rx_0$. Also $F^sX \subset TX$. We can also choose a point x_2 in X such that $Tx_2 = F^sx_1$. In general $Tx_{2n+1} = E^rx_{2n+1}$, $Tx_{2n+2} = F^sx_{2n+1}$ for $n = 0, 1, 2, \cdots$

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2n+2}) &= d(E^rx_{2n}, F^sx_{2n+1}) \\ &\leq \frac{K.d(Tx_{2n},\ E^rx_{2n}).d(Tx_{2n+1}, F^sx_{2n+1})}{d(Tx_{2n+1}, F^sx_{2n+1}) + d(Tx_{2n+1}, E^rx_{2n})} \\ &\leq \frac{K.d(Tx_{2n},\ Tx_{2n+1}).d(Tx_{2n+1}, Tx_{2n+2})}{d(Tx_{2n+1}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+1})} \\ &\leq K.d(Tx_{2n},\ Tx_{2n+1}) \end{aligned}$$

$$\Rightarrow d(Tx_{2n+1}, Tx_{2n+2}) \leq Kd(Tx_{2n},\ Tx_{2n+1})$$

Similarly we can see $d(Tx_{2n}, Tx_{2n+1}) \leq K\ d(Tx_{2n-1}Tx_{2n})$

Proceeding in this way, we have $d(Tx_{2n+1}, Tx_{2n+2}) \leq K^{2n+1}\ d(Tx_0, Tx_1)$.

By routine calculation the following inequalities holds for $q > 0$

$$d(Tx_n, Tx_{n+q}) \leq \sum_{i=1}^{q} d(Tx_{n+i-1}, Tx_{n+i})$$

$$d(Tx_n, Tx_{n+q}) \leq K^{n+1}\ d(Tx_0, Tx_1) \to 0 \text{ as } n \to \infty \text{ (since } K < 1)$$

Hence $\{Tx_n\}$ is a Cauchy sequence. By the completeness of X, $\{Tx_n\}$ converge to a point p in X. From (2) $\{E^rx_{2n}\}$ and $\{F^sx_{2n+1}\}$ are subsequences of $\{Tx_n\}$ also converge to the same point p in X.

Now $E^rTx_{2n} = TE^rx_{2n} \to Tp$, $F^sTx_{2n+1} = TF^sx_{2n+1} \to Tp$, $TTx_n \to Tp$ (3)

Now we prove $E^rp = Tp$ and $F^sp = Tp$ using (c), (1) and (3)

$$d(E^rp, Tp) = d(E^rp, TF^sx_{2n+1})$$

$$= d(E^r p, F^s T x_{2n+1})$$

$$\leq \frac{K\ d(Tp, E^r p)\ d(TTx_{2n+1}, F^s T x_{2n+1})}{d(TTx_{2n+1}, F^s x_{2n+1}) + d(Tx_{2n+1}, E^r p)}$$

Since $TTx_{2n+1} \to Tp$ and $F^s T x_{2n+1} \to Tp$ by (3). Thus we get $E^r p = Tp$ as $n \to \infty$. Similarly we can see $F^s p = Tp$ as $n \to \infty$. Thus we get $E^r p = Tp = F^s p$.

Also

$$d(Tp, p) = d(TE^r x_{2n}, F^s x_{2n+1})$$

$$= d(E^r T x_{2n}, F^s x_{2n+1})$$

$$\leq \frac{d(TTx_{2n}, E^r T x_{2n})\ d(Tx_{2n+1}, F^s x_{2n+1})}{d(Tx_{2n+1}, F^s x_{2n+1}) + d(Tx_{2n+1}, E^r T x_{2n})}$$

Since $TTx_{2n} \to Tp$ and $E^r T x_{2n} \to Tp$ by (3). Thus we get $Tp = p$ as $n \to \infty$.

Now

$$E^r = Tp = F^s p = p \Rightarrow E^r p = p \Rightarrow E(E^r p) = Ep \tag{4}$$

Also

$$Tp = p \Rightarrow ETp = Ep \Rightarrow TEp = Ep \tag{5}$$

i.e. from (4) and (5) $E(E^r p) = Ep = T(Ep)$ i.e. $E^r(Ep) = Ep = T(Ep)$.

i.e. Ep is the common fixed point of E^r and T. Similarly Fp is the common fixed point of T and Fp. But p is unique common fixed point of E^r, F^s and T. Hence $Ep = Tp = p = Fp$, uniqueness of p is trivial.

Theorem 2. Let E, F and T be three self maps of a complete metric space (X, d) satisfying the following conditions:

(a) $\{E, T\}$ and $\{F, T\}$ are commuting pairs

(b) $EX \subset TX$, $FX \subset TX$.

(c) There exist integers $r, s > 0$ such that

$$d(E^r x, F^s y) \leq K \frac{d(Tx, E^r x)(Ty, F^s y)}{d(Tx, F^s y) + d(Ty, E^r x) + d(Tx, Ty)}$$

for every $x, y \in X$ and $0 < K < 1$.

Then E, F and T have a unique common fixed point in X, provided T is continuous.

Proof. using (a) and (b)

$$E^r T = TE^r, \quad F^s T = TF^s \tag{6}$$

and

$$E^r X \subset EX \subset TX, \quad F^s T \subset FX \subset TX \tag{7}$$

Let x_0 be any arbitrary point in X. Since $E^r X \subset TX$. We can choose a point x_1 in X such that $Tx_1 = E^r x_0$. Also $F^s X \subset TX$, We can choose a point x_2 in X such that $Tx_2 = F^s x_1$. In general $Tx_{2n+1} = E^r x_{2n}$, $Tx_{2n+2} = F^s x_{2n+1}$, for $n = 0, 1, 2, \cdots$

Now we consider

$$d(T_{2n+1}, Tx_{2n+2}) = d(E^r x_{2n}, F^s x_{2n+1})$$

$$\leq \frac{Kd(Tx_{2n}, E^r x_{2n})d(Tx_{2n+1}, F^s x_{2n+1})}{d(Tx_{2n}, F^s x_{2n+1}) + d(Tx_{2n+1}, E^r x_{2n}) + d(x_{2n}, Tx_{2n+1})}$$

$$\leq K\, d(Tx_{2n}, Tx_{2n+1})$$

$$\Rightarrow d(Tx_{2n+1}, Tx_{2n+2}) \leq Kd(Tx_{2n}, Tx_{2n+1})$$

Similarly we can see $d(Tx_{2n}, Tx_{2n+1}) \leq K\, d(Tx_{2n-1}, Tx_{2n})$

Proceeding in this way, we have $d(Tx_{2n}, Tx_{2n+1}) \leq K^{2n+1}\, d(Tx_0, Tx_1)$

By routine calculation the following inequalities holds for $p > 0$

$$d(Tx_n, Tx_{n+p}) \leq \sum_{i=1}^{p} d(Tx_{n+i-1}, Tx_{n+i})$$

$$\Rightarrow d(Tx_n, Tx_{n+p}) \leq \frac{K^{n+i-1}}{1-K}\, d(Tx_0, Tx_1) \to 0 \text{ as } n \to \infty \text{ (since } K < 1)$$

Hence $\{Tx_n\}$ is a Cauchy sequence. By the completeness of X, $\{Tx_n\}$ converges to a point u in X. From (7), $\{E^r x_{2n}\}$ and $\{F^s x_{2n+1}\}$ are subsequence of $\{Tx_n\}$, also converge to the same point u in X.

Now

$$\begin{aligned} E^rTx_{2n} &= TE^rx_{2n} \to Tu \\ F^sTx_{2n+1} &= TF^sx_{2n+1} \to Tu \\ TTx_n &\to Tu \end{aligned} \tag{8}$$

Now we prove $E^ru = Tu$ and $F^su = Tu$ using (7), (c) and (8)

$$\begin{aligned} d(E^ru, Tu) &= d(E^ru, TF^sx_{2n+1}) \\ &= d(E^ru, F^sTx_{2n+1}) \\ &\leq \frac{K\ d(Tu, E^ru)\ d(TTx_{2n+1}, F^sTx_{2n+1})}{d(Tu, F^sTx_{2n+1}) + d(TTx_{2n+1}, E^ru) + d(Tu, TTx_{2n+1})} \end{aligned}$$

Since $TTx_{2n+1} \to Tu$, and $F^sTx_{2n+1} \to Tu$ by (8). Thus we get $E^ru = Tu$ as $n \to \infty$. Similarly we can see $F^su = Tu$ as $n \to \infty$. Thus we get $E^ru = Tu = F^su$.

Also

$$\begin{aligned} d(Tu, u) &= d(E^rx_{2n}, F^sx_{2n+1}) \\ &= d(E^rTx_{2n}, F^sx_{2n+1}) \\ &\leq \frac{Kd(TTx_{2n}, E^rTx_{2n})\ d(Tx_{2n+1}, F^sx_{2n+1})}{d(TTx_{2n}, F^sx_{2n+1}) + d(Tx_{n+1}, E^rTx_{2n}) + d(TTx_{2n}, Tx_{2n+1})} \end{aligned}$$

Since $TTx_{2n} \to Tu$ and $E^rTx_{2n} \to Tu$ by (3). Thus we get $Tu = u$ as $n \to \infty$.

Now

$$E^ru = Tu = F^su = u \Rightarrow E^ru = u \Rightarrow E(E^ru) = Eu \tag{9}$$

Also

$$Tu = u \Rightarrow ETu = Eu \Rightarrow TEu = Eu \tag{10}$$

i.e. from (9) and (10) $E(E^ru) = Eu = T(Eu)$ i.e. $E^r(Eu) = Eu = T(Eu)$.

i.e. Eu is the common fixed point of E^r and T. Similarly Fp is the common fixed point of T and Fu. But u is unique common fixed point of E^r, F^s and T. Hence $Eu = Tu = u = Fu$. Uniqueness of u is trivial.

Corollary. Let E, F and T be three self maps of a complete metric space (X, d) such that T is continuous and E, F, T satisfy the conditions

(a) $\{E, T\}$ and $\{F, T\}$ are commuting pairs.

(b) $EX \subset TX$, $FX \subset TX$.

(c) There exist integers $r, s > 0$ such that

$$d(E^r x, F^s y) \ \leq \ K \frac{d(Tx, E^r x)(Ty, F^s y)}{d(Tx, F^s y) + d(Tx, Ty)}$$

for every $x, y \in X$ and $0 < K < 1$. Then E, F and T have a unique common fixed point in X.

Proof. Proof is obvious.

References

[1] Jaggi, D. S. and Das, B. K., *An extension of Banach fixed point theorem through a rational expression*, Bull. Cal. Math. Soc. **72** (1980), 261.

[2] Murthy, P. P. and Sharma, B. K., *A fixed point theorem for three mappings*, Bull. Cal. Math. Soc. **83** (1991), 371-372.