

GENERATING FUNCTIONS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY GENERALIZED HERMITE POLYNOMIALS

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Abstract: In present paper, we obtain the generating function and bilateral generating function for biorthogonal polynomials suggested by the generalized Hermite polynomials.

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1. Introduction

Recently Andhare and Jagtap [2] constructed a pair of biorthogonal polynomials suggested by generalized Hermite polynomials.

$$\begin{aligned} S_{2n}(x; k, l) &= \frac{n! (-1)^n 2^{2n} (1/2)_n Z_n^\beta(x^{2k}; l)}{(1 + \beta)_n} \\ &= \frac{2^{2n} (1/2)_n \Gamma(ln + \beta + 1)}{(1 + \beta)_n} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{2kln-2klj}}{\Gamma(ln - lj + \beta + 1)} \end{aligned} \quad (1.1)$$

$$\begin{aligned} S_{2n+1}(x; k, l) &= \frac{(-1)^n n! (3/2)_n 2^{2n+1} x^l z_n^{-\beta l}(x^{2k}; l)}{(1 - \beta)_n} \\ &= \frac{(3/2)_n 2^{2n+1} \Gamma(ln - \beta l + 1)}{(1 - \beta)_n} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{2kln-2klj+l}}{\Gamma(ln - lj + \beta l + 1)} \end{aligned} \quad (1.2)$$

and

$$T_{2n}(x; k, l) = \frac{(-1)^n n! (1 + n)_n}{(1 + \beta)_n} Y_n^\beta(x^{2k}; l)$$

$$= \frac{(-1)^n 2^{2n} (1/2)_n}{(1+\beta)_n} \sum_{r=0}^n \frac{x^{2kn-2kr}}{(n-r)!} \sum_{s=0}^{n-r} (-1)^s \binom{n-r}{s} \left(\frac{s+\beta+1}{l} \right)_n \quad (1.3)$$

$$\begin{aligned} T_{2n+1}(x; k, l) &= \frac{(-1)^n n! 2^{2n+1} (3/2)_n}{(1-\beta)_n} x Y_n^{-\beta l}(x^{2k}; l) \\ &= \frac{(-1)^n n! 2^{2n+1} (3/2)_n}{(1-\beta)_n} \sum_{r=0}^n \frac{x^{2kn-2kr+1}}{(n-r)!} \\ &\quad \sum_{s=0}^{n-r} (-1)^s \binom{n-r}{s} \left(\frac{s-\beta l+1}{l} \right)_n \end{aligned} \quad (1.4)$$

Combining (1.1) and (1.2) we obtain first set of biorthogonal polynomials $S_n(x; k, l)$ as

$$\begin{aligned} S_n(x; k, l) &= \frac{2^n ((1+2\epsilon)/2)_{[n/2]} \Gamma((ln+2\beta-\epsilon(2\beta+2\beta l+l))/2+1)}{(1+\beta-2\beta\epsilon)_{[n/2]}} \\ &\quad \times \sum_{j=0}^{[n/2]} (-1)^j \binom{[n/2]}{j} \frac{x^{kln-2klj+l(1-k)\epsilon}}{\Gamma((ln+2\beta-\epsilon(2\beta+2\beta l+l))/2+1-lj)}. \end{aligned} \quad (1.5)$$

Similarly combining (1.3) and (1.4) we get second set of biorthogonal polynomials $T_n(x; k, l)$ as

$$\begin{aligned} T_n(x; k, l) &= \frac{(-1)^{[n/2]} 2^n ((1+2\epsilon)/2)_{[n/2]}}{(1+\beta-2\beta\epsilon)_{[n/2]}} \sum_{r=0}^{[n/2]} \frac{x^{kn-2kr+(1-k)\epsilon}}{([n/2]-r)!} \\ &\quad \times \sum_{s=0}^{[n/2]-r} (-1)^s \binom{[n/2]-r}{s} \left(\frac{s+\beta+1-(\beta+\beta l)\epsilon}{l} \right)_{[n/2]} \end{aligned} \quad (1.6)$$

where ϵ is 0 or 1 according to n is even or odd integer respectively and n is greatest integer less than or equal to n .

For $\beta = -1/2k$, $k = 1$ the polynomials (1.1), (1.2), (1.3) and (1.4) reduces to putting $\mu = 0$ in the polynomials defined by Thakare and Madhekar [8,9].

$$S_{2n}(x; 1, l) = \frac{\Gamma(nl+1/2)}{\Gamma(nl+k/2)} S_{2n}^\mu(x; l) = \frac{\Gamma(nl+1/2)}{\Gamma(nl+k/2)} S_{2n}(x, l). \quad (1.7)$$

$$S_{2n+1}(x; 1, l) = S_{2n+1}^\mu(x; l) = S_{2n+1}(x, l). \quad (1.8)$$

$$T_{2n}(x; 1, l) = T_{2n}^\mu(x; l) = T_{2n}(x, l) \quad (1.9)$$

$$T_{2n+1}(x; 1, l) = T_{2n+1}^\mu(x; l) = T_{2n+1}(x, l) \quad (1.10)$$

For $l = 1$ the above equation reduces to even and odd Hermite polynomials respectively.

2. Generating Functions

We have generating function for $Z_n^\alpha(x; k)$ obtained by Srivastava [7, p.490] (see also Karande and Thakare [5])

$$\sum_{n=0}^{\infty} \frac{Z_n^\alpha(x; k)t^n}{(1+\alpha)_{kn}} = e^t {}_0F_k \left[\begin{matrix} - & ; \\ \Delta(k, l+\alpha) & ; \end{matrix} - (x/k)^k t \right]. \quad (2.1)$$

Again we recall the generating function obtained by Karande and Thakare [5]

$$\sum_{n=0}^{\infty} \frac{(c)_n Z_n^\alpha(x; k)t^n}{(1+\alpha)_{kn}} = (1-t)^{-c} {}_1F_k \left[\begin{matrix} c & ; \\ \Delta(k, l+\alpha) & ; \end{matrix} \frac{-tx^k}{k^k(1-t)} \right] \quad (2.2)$$

By equations (1.1) and (2.2), we obtain

$$\sum_{n=0}^{\infty} \frac{(c)_n(1+\beta)_n S_{2n}(x; k, l)t^{2n}}{(1/2)_n(1+\beta)_{nl}n!} = (1+4t^2)^{-c} {}_1F_l \left[\begin{matrix} c & ; \\ \Delta(l, l+\beta) & ; \end{matrix} \frac{4t^2 x^{2kl}}{l^l(1+4t^2)} \right] \quad (2.3)$$

From equations (1.2) and (2.2), we obtain

$$\sum_{n=0}^{\infty} \frac{(c)_n(1-\beta)_n(1+2n)S_{2n+1}(x; k, l)t^{2n+1}}{(1-\beta l)_{nl}(3/2)_n n!} = t(1+\theta) \sum_{n=0}^{\infty} \frac{(c)_n(1-\beta)_n S_{2n+1}(x; k, l)t^{2n}}{(1-\beta l)_{nl}(3/2)_n n!} \quad (2.4)$$

where $\theta = t \frac{d}{dt}$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(c)_n(1-\beta)_n S_{2n+1}(x; k, l)t^{2n+1}(2n+1)}{(1-\beta l)_{nl}(3/2)_n n!} \\ &= \frac{2tx^l(1+4t^2-8ct^2)}{(1+4t^2)^{c+1}} {}_1F_l \left[\begin{matrix} c & ; \\ \Delta(l, l-\beta l) & ; \end{matrix} \frac{4t^2 x^{2kl}}{l^l(1+4t^2)} \right] \end{aligned}$$

$$+ \frac{16ct^3 x^{2kl+l}}{(1-\beta l)_l (1+4t^2)^{c+2}} {}_1F_l \left[\begin{matrix} c+1 & ; \\ \Delta(l, 1-\beta l+1) & ; \end{matrix} \frac{4t^2 x^{2kl}}{l^l (1+4t^2)} \right] \quad (2.5)$$

Combining (2.3) and (2.5) we get the generating function for $S_n(x; k, l)$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(c)_{[n/2]} (1+2n\epsilon) (1+\beta-2\beta\epsilon)_{[n/2]} S_n(x; k, l) t^n}{(1/2+\epsilon)_{[n/2]} (1+\beta-\beta(l+1)\epsilon)_l [n/2]!} \\ &= (1+4t^2)^{-c} {}_1F_l \left[\begin{matrix} c & ; \\ \Delta(l, 1+\beta) & ; \end{matrix} \frac{4t^2 x^{2kl}}{l^l (1+4t^2)} \right] \\ &+ \frac{2tx^l (1+4t^2-8ct^2)}{(1+4t^2)^{c+1}} {}_1F_l \left[\begin{matrix} c & ; \\ \Delta(l, 1-\beta l) & ; \end{matrix} \frac{4t^2 x^{2kl}}{l^l (1+4t^2)} \right] \\ &+ \frac{16ct^3 x^{2kl+l}}{(1-\beta l)_l (1+4t^2)^{c+2}} {}_1F_l \left[\begin{matrix} c+1 & ; \\ \Delta(l, 1-\beta l+l) & ; \end{matrix} \frac{4t^2 x^{2kl}}{l^l (1+4t^2)} \right] \end{aligned} \quad (2.6)$$

Particular cases

For $k=1$, $l=k$ equation (2.6) reduces to

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(c)_{[n/2]} (1+2n\epsilon) (1+\beta-2\beta\epsilon)_{[n/2]} S_n(x; k) t^n}{(1/2+\epsilon)_{[n/2]} (1+\beta-\beta(k+1)\epsilon)_k [n/2]!} \\ &= (1+4t^2)^{-c} {}_1F_k \left[\begin{matrix} c & ; \\ \Delta(k; 1+\beta) & ; \end{matrix} Z \right] + \frac{2x^k (1+4t^2-8ct^2)}{(1+4t^2)^{c+1}} {}_1F_k \left[\begin{matrix} c & ; \\ \Delta(k, 1-\beta k) & ; \end{matrix} Z \right] \\ &+ \frac{16ct^3 x^{3k}}{(1-\beta k)_k (1+4t^2)^{c+2}} {}_1F_k \left[\begin{matrix} c+1 & ; \\ \Delta(k, 1-\beta k+k) & ; \end{matrix} Z \right] \end{aligned} \quad (2.7)$$

where $Z = \frac{4t^2 x^{2k}}{k^k (1+4t^2)}$.

For $k=1$ and $\beta = -1/2$ equation (2.7) reduces to generating functions for Hermite polynomials first obtained by Brafman [3, p.949, (37)].

3. Bilateral Generating Functions

Let

$$M_{n,q}^{p,\mu}(y_1, y_2, \dots, y_N; z) = \sum_{j=0}^{[n/q]} \frac{1}{(n-qj)} C_j \Omega_{\mu+pj}(y_1, y_2, \dots, y_N) Z^j \quad (3.1)$$

$$\Lambda_{m,q}(x; y_1, y_2, \dots, y_N; t) = \sum_{n=0}^{\infty} C_n T_{m+qn}(x; k) \Omega_{m+qn}(y_1, y_2, \dots, y_N) t^n \quad (3.2)$$

where μ is an arbitrary complex number, p, q are positive integers and $\Omega_{\mu}(y_1, y_2, \dots, y_N)$ is non-vanishing functions of N variables $y_1, y_2, \dots, y_N$, $N \geq 1$, $C_j \neq 0$, $j = 1, 2, 3, \dots$

Consider

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{T_{2m+2n}(x; k, l) M_{n,q}^{p,\mu}(y_1, y_2, \dots, y_N; z) t^{2n} (1+\beta)_{m+n}}{(1/2)_{m+n}} \\ &= \sum_{n=0}^{\infty} T_{2m+2n}(x; k, l) \sum_{j=0}^{[n/q]} \frac{1}{(n-qj)!} C_j \frac{\Omega_{\mu+pj}(y_1, y_2, \dots, y_N)}{(1/2)_{m+n}} Z^j t^{2n} (1+\beta)_{m+n}. \end{aligned} \quad (3.3)$$

Using (1.3) in R.H.S. of (3.3) we get (3.4) L.H.S. of (3.3)

$$\begin{aligned} &= \sum_{j=0}^{\infty} C_j \Omega_{\mu+pj}(y_1, y_2, \dots, y_N) (m+qj)! (-1)^{m+qj} 2^{2m+2qj} \\ & \quad (Zt^{2q})^j \sum_{n=0}^{\infty} \binom{m+n+qj}{n} Y_{m+n+qj}^{\beta}(x^{2k}; l) (-4t^2)^n. \end{aligned} \quad (3.4)$$

From the generating function obtained by Parbhakar [6] (see also Srivastava [7]).

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha}(x^2; k) t^n &= (1-t)^{-(\alpha+m+1)/k} \exp\{x^2[1-(1-t)^{-1/k}]\} \\ & Y_m^{\alpha}(x^2(1-t^2)^{-1/k}; k) \end{aligned} \quad (3.5)$$

where m is any non-negative integer.

Due to (3.5), (3.4) reduces to

$$\sum_{n=0}^{\infty} T_{2m+2n}(x; k, l) M_{n,q}^{p,\mu}(y_1, y_2, \dots, y_N; z) t^{2n} \frac{(1+\beta)_{m+n}}{(1/2)_{m+n}}$$

$$\begin{aligned}
&= \exp\{x^{2k}[1 - (1 + 4t^2)^{-1/l}]\}(1 + 4t^2)^{-(\beta+ml+1)/l} \sum_{j=0}^{\infty} \frac{(1 + \beta)_{m+q_j}}{(1/2)_{m+q_j}} C_j \\
&\Omega_{\mu+p_j}(y_1, y_2, \dots, y_N) T_{2m+2q_j}(x(1 + 4t^2)^{-(1/2kl)}; k, l) \left[Z \left(\frac{t^2}{1 + 4t^2} \right)^q \right]^j \quad (3.6)
\end{aligned}$$

This is bilateral generating function for even case of second set $\{T_n(x; k, l)\}$.

Similarly we obtain bilateral generating function for the odd case of second set $\{T_n(x; k, l)\}$ as follows

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{2m+2n+1}(x; k, l) \frac{(1 - \beta)_{m+n}}{(3/2)_{m+n}} M_{m,q}^{p,\mu}(y_1, y_2, \dots, y_N; z) t^{2n+1} \\
&= t(1 + 4t^2)^{-(-\beta l - (1/2k) + ml + 1)/l} \exp\{x^{2k}[1 - (1 + 4t^2)^{-1/l}]\} \sum_{j=0}^{\infty} C_j \frac{(1 - \beta)_{m+q_j}}{(3/2)_{m+q_j}} \\
&\Omega_{\mu+p_j}(y_1, y_2, \dots, y_N) T_{2m+2q+1}(x(1 + 4t^2)^{-(1/2kl)}; k, l) \left[Z \left(\frac{t^2}{1 + 4t^2} \right)^q \right]^j \quad (3.7)
\end{aligned}$$

Combining equations (3.6) and (3.7), we obtain bilateral generating function.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{T_{2m+n}(x; k, l) M_{[n/2],q}^{p,\mu}(y_1, y_2, \dots, y_N; z) (1 + \beta - 2\beta\epsilon)_{m+[n/2]} t^n}{(1/2 + \epsilon)_{m+[n/2]}} \\
&= \exp\{x^{2k}[1 - (1 + 4t^2)^{-1/l}]\} \sum_{j=0}^{\infty} C_j \Omega_{\mu+p_j}(y_1, y_2, \dots, y_N) \left[Z \left(\frac{t^2}{1 + 4t^2} \right)^q \right]^j \\
&\left\{ \frac{[(1 + 4t^2)^{-(-\beta+ml+1)/l} (1 + \beta)_{m+q_j} T_{2m+2q_j}(x(1 + 4t^2)^{-(1/2kl)}; k, l)]}{(1/2)_{m+q_j}} \right. \\
&\left. + \frac{t(1 + 4t^2)^{-(-\beta l - (1/2k) + ml + 1)/l} (1 - \beta)_{\mu+q_j} T_{2m+2q_j+1}(x(1 + 4t^2)^{-(1/2kl)}; k, l)}{(3/2)_{m+q_j}} \right\} \quad (3.8)
\end{aligned}$$

Putting $\beta = -1/2$, $k = 1$, $l = k$ in (3.6) and using definition of (1.3), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{2m+2n}(x; k) M_{n,q}^{p,\mu}(y_1, y_2, \dots, y_N; z) t^{2n} = (1 + 4t^2)^{-(-mk+1/2)/k} \\
&\exp\{x^2[1 - (1 + 4t^2)^{-1/k}]\} \Lambda_{2m,2q} \left[x(1 + 4t^2)^{-(1/2k)}; y_1, y_2, \dots, y_N Z \left(\frac{t^n}{1 + 4t^2} \right)^q \right] \quad (3.9)
\end{aligned}$$

Similarly putting $\beta = -1/2$, $k = 1$, $l = k$ in (3.7) and using definition of (1.4), we get

$$\sum_{n=0}^{\infty} T_{2m+2n+1}(x; k) M_{n,q}^{p,\mu}(y_1, y_2, \dots, y_N; z) t^{2n+1} = t(1 + 4t^2)^{-(mk+k/2+1/2)/k}$$

$$\times \exp\{x^2[1-(1+4t^2)^{-1/k}]\} \Lambda_{2m+1,2q} \left[x(1+4t^2)^{-1/2k}; y_1, y_2, \dots, y_N : Z \left(\frac{t^2}{1+4t^2} \right)^q \right]. \quad (3.10)$$

Combining (3.9) and (3.10), we obtain a generating function given by Andhare and Jagtap [1, p.86].

$$\begin{aligned} \sum_{n=0}^{\infty} T_{2m+n}(x; k) M_{[n/2],q}^{p,\mu}(y_1, y_2, \dots, y_N; z) t^n &= (1+4t^2)^{-(mk+(k+1)/2)/k} \\ &\cdot \exp\{x^2[1-(1+4t^2)^{-1/k}]\} \left\{ (1+4t^2)^{1/2} \Lambda_{2m,2q} \left[x(1+4t^2)^{-1/2k}; y_1, y_2, \dots, y_N ; Z \left(\frac{t^2}{1+4t^2} \right)^q \right] \right. \\ &\quad \left. + t \Lambda_{2m+1,2q} \left[x(1+4t^2)^{-1/2k}; y_1, y_2, \dots, y_N ; Z \left(\frac{t^2}{1+4t^2} \right)^q \right] \right\} \end{aligned} \quad (3.11)$$

Particular Cases

Case I : For $k = 1$, equation (3.11) gives the bilateral generating function for Hermite polynomials.

$$\begin{aligned} \sum_{n=0}^{\infty} H_{2m+n}(x) M_{[n/2],q}^{p,\mu}(y_1, y_2, \dots, y_N; z) t^n &= (1+4t^2)^{-(m+1)} \exp \left\{ \frac{4x^2 t^2}{1+4t^2} \right\} \\ &\left\{ (1+4t^2)^{1/2} \Lambda'_{2m,2q} \left[x(1+4t^2)^{-1/2}; y_1, y_2, \dots, y_N ; Z \left(\frac{t^2}{1+4t^2} \right)^q \right] \right. \\ &\quad \left. + t \Lambda'_{2m+1,2q} \left[x(1+4t^2)^{-1/2}; y_1, y_2, \dots, y_N ; Z \left(\frac{t^2}{1+4t^2} \right)^q \right] \right\} \end{aligned} \quad (3.12)$$

where

$$\Lambda'_{m,q}[x; y_1, y_2, \dots, y_N; t] = \sum_{j=0}^{\infty} C_n H_{m+q_n}(x) \Omega_{\mu+p_j}(y_1, y_2, \dots, y_N) t^n$$

Case II : By substituting $m = 0$, $q = 1$, $C_j \Omega_{\mu+p_j}(y_1, y_2, \dots, y_N) = \frac{\delta_j}{j!}$ we obtain the following bilateral generating function due to Thakare and Madhekar [8, p.36(4.6)].

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{T_n(x; k) \sigma_n(z) t^n}{[n/2]!} &= (1+4t^2)^{-(k+1)/2k} \\ &\cdot \exp\{x^2[1-(1+4t^2)^{-1/k}]\} G \left(X ; \frac{zt^2}{1+4t^2} \right) \end{aligned} \quad (3.13)$$

where $\sigma_n(z) = \sum_{j=0}^{[n/2]} \binom{[n/2]}{j} \delta_j z^j$, $X = x(1 + 4t^2)^{-1/2k}$

and

$$G(x; z) = \sum_{n=0}^{\infty} \delta_n (1 + 4t^2)^{1/2} T_{2n}(x; k) + T_{2n+1}(x; k) \frac{z^n}{n!} \quad (3.14)$$

Case III : For $z = 0$ $\delta_0 = 1$ in equation (3.14), we get the generating function obtained by Thakare and Madhekar [8]

$$\sum_{n=0}^{\infty} \frac{T_n(x; k) t^n}{[n/2]!} = Y(1 + 4t^2)^{-(k+2)/2k} [2xt + (1 + 4t^2)^{(k+1)/2k}] \quad (3.15)$$

where $Y = \exp\{x^2[1 - (1 + 4t^2)^{-1/k}]\}$.

Case IV : For $k = 1$, equation (3.15) reduces to the generating function for Hermite polynomials given by Doetsch [4].

$$\sum_{n=0}^{\infty} \frac{H_n(x) t^n}{[n/2]!} = (1 + 4t^2)^{-3/2} (1 + 2xt + t^2) \exp\left\{\frac{4x^2 t^2}{1 + 4t^2}\right\}. \quad (3.16)$$

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