

EXISTENCE OF SOLUTIONS FOR GENERALIZED VECTOR VARIATIONAL-LIKE INEQUALITIES¹

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Abstract : In this paper, we establish some existence theorems for a new class of generalized vector variational-like inequalities in Banach space and in Hausdörff topological vector space under compact and noncompact assumptions on underlying convex sets.

Keywords : Generalized vector variational-like inequalities, escaping sequences, upper semicontinuous affine and pseudomonotone mappings.

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1. Introduction

The Vector variational inequality problem was first introduced and studied by Giannessi [6] in finite dimensional Euclidean spaces. Later, Chen *et.al.* [1-3], Lee *et.al* [9], Siddiqi *et.al* [13] and Yang [16], established some kinds of vector variational inequalities and vector variational-like inequalities in general Banach spaces and Hausdörff topological vector spaces. Recently, Kazmi and Khan [7] have intensively studied and established some existence theorems for solutions of some new class of generalized vector variational-like inequalities in reflexive Banach spaces.

Motivated and inspired by the recent work on vector variational inequalities and vector variational-like inequalities in this paper we consider a new class of generalized vector variational-like inequality problem (in short GVVLIP) of which mostly previously known vector variational (variational-like) inequality problems are special cases. Further, using Fixed point Theorem [15] and a theorem due to Fan [5], we establish some existence results for GVVLIP in Hausdörff topological vector space and Banach space under compact and noncompact assumptions on

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underlying convex sets. The use of the escaping sequence is crucial for the proof of some existence results established in this paper.

2. Preliminaries

Let X, Y be Hausdörff topological vector spaces; Z be an ordered Hausdörff topological vector space and Q be a closed, convex, pointed, proper and solid cone in Z . Then an ordering ' $\leq$ ' (weak ordering $<$) in Q is defined by means of

$$x \leq 0 \Leftrightarrow x \in -Q, \quad x < 0 \Leftrightarrow x \in -\text{int}Q.$$

where $\text{int}Q$ denotes the interior of the set Q .

This notation can be extended to arbitrary set $P \subset Z$ by setting,

$$P \leq 0 \Leftrightarrow P \subseteq -Q, \quad P < 0 \Leftrightarrow P \subseteq -\text{int}Q.$$

Following Yu [17], a point x_0 is called a vector maximum point of $A \subseteq Z$ if the set $\{x_1 \in A : x_0 \leq x_1, x_1 \neq x_0\} = \emptyset$ which is equivalent to

$$A \cap (x_0 + Q) = \{x_0\}.$$

We note that if A be a non empty and compact set in Z . Then $\text{Max}(A) \neq \emptyset$, see Luc [10]. Let K and C be non-empty subsets of X and Y respectively. Let $M : K \times C \rightarrow 2^{L(X,Z)}$, where $L(X, Z)$ is the space of all linear continuous mappings, $T : K \rightarrow 2^C$ be a multivalued mapping and $\eta : K \times K \rightarrow X$ be a non-linear mapping. Let $\{P(x) : x \in K\}$ be a family of closed, pointed and convex solid cones in Z with $\text{int}P(x) \neq \emptyset, \forall x \in K$.

Now, we consider the following generalized vector variational-like inequality problem (GVVLIP). Find $x_0 \in K$ such that for each $y \in K, \exists u_0 \in T(x_0)$ and

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subseteq Z \setminus (-\text{int}P(x_0)). \quad (2.1)$$

Some Special Cases

1. Let $X = R^n, Y = R^m, Z = R^l, P(x) = R_+^l \forall x \in K$, and let $L : K \times C \rightarrow R^l$. If $M(x_0, u_0) = L'(x_0, u_0)$, where L' denotes the Frechet derivative of L at x_0 , and $T : K \rightarrow 2^C$ is defined by

$$T(x_0) := \{y \in C : L(x_0, z) - L(x_0, y) \notin \text{int}R_+^l, \quad \forall z \in C\},$$

then GVVLIP reduces to the problem of finding of $x_0 \in K$ such that for each $y \in K, \exists u_0 \in T(x_0)$ and

$$\langle L'(x_0, u_0), \eta(y, x_0) \rangle \notin -\text{int}R_+^l \quad (2.2)$$

which has application in establishing the existence of global weak saddle point of the vector valued mapping L , see for example Kazmi [8].

2. If $Y = Z$, $M(x_0, u_0) = A(x_0)$ and $\eta(y, x) = y - g(x)$, where $g : K \rightarrow K$ and $A : K \rightarrow L(X, Y)$ are single-valued mappings, then GVVLP reduces to finding $x_0 \in K$, such that

$$\langle A(x_0), y - g(x_0) \rangle \notin -\text{int}P(x_0), \quad \forall y \in K, \quad (2.3)$$

which is the GVVI problem due to Siddiqi *et.al*[13].

3. Let $Z = Y = R$, $M(x_0, u_0) = A(x_0)$, $P(x) = R^+$, $L(X, Y) = X^*$ (the dual of X) then GVVLP reduces to finding $x_0 \in K$ such that

$$\langle A(x_0), \eta(x_0, y) \rangle \geq 0, \quad \forall y \in K, \quad (2.4)$$

which is the VLI problem introduced and studied by Siddiqi *et.al* [14] in infinite dimensional spaces.

We remark that GVVLP also reduces to the problems considered in [2,5,12]. Next we recall the following concepts and results which are important for the proof of our main results.

Definition 2.1. Let X be a topological vector space. A map $F : X \rightarrow 2^X$ is called KKM map, if the convex hull of every finite subset $\{x_1, x_2, x_3, \dots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$.

Theorem 2.1[15]. Let K be a nonempty and convex subset of Hausdörff topological vector space X , and $S : K \rightarrow 2^K$ a multivalued mapping such that

- (i) For each $x \in K$ $S(x)$ is a nonempty convex subset of K ;
- (ii) For each $x \in K$, $S^{-1}(y) := \{x \in K : y \in S(x)\}$ contains a relatively open subset O_y of K (O_y may be empty for some $y \in K$);
- (iii) $\bigcup_{y \in K} O_y = K$;
- (iv) K contains a nonempty subset K_0 which is contained in a compact convex subset of K_1 of K such that the set $D = \bigcup_{y \in K_0} O_y^c$ is either empty or compact, where O_y^c denotes the complement of O_y in K .

Then $\exists$ a point $x_0 \in S(x_0)$.

Theorem 2.2[5]. Let E be a nonempty, compact convex set of a Hausdörff topological vector space, let and A be a subset of $E \times E$ having the following properties;

- (i) $(x, x) \in A, \forall x \in E$;
- (ii) $\forall x \in E, A_x := \{y \in E : (x, y) \in A\}$ is closed in E ;
- (iii) $\forall y \in E, A_y := \{x \in E : (x, y) \notin A\}$ is convex.

Then $\exists y_0 \in E$, such that $E \times \{y_0\} \subseteq A$.

Definition 2.2. Let $F : X \rightarrow 2^Y$ be a multivalued mapping from a normed space X to a normed space Y , is convex if and only if for all $x_1, x_2 \in \text{Dom}(F)$, for all $\alpha \in [0, 1]$,

$$\alpha F(x_1) + (1 - \alpha)F(x_2) \subset F(\alpha x_1 + (1 - \alpha)x_2).$$

Definition 2.3. Let X, Y and Z be three Hausdörff topological vector spaces and $K \subseteq X$ and $C \subset Y$. Let $M : K \times C \rightarrow 2^{L(X, Z)}, T : K \rightarrow 2^C$ be multivalued mappings and $\eta : K \times K \rightarrow X$ be a mapping and let $P = \bigcap_{x \in K} P(x) \neq \emptyset$. Then M is said to be:

- (a) η -monotone in C with respect to T , if for $x, y \in X, u \in T(x)$ and $v \in T(y)$ such that

$$\langle M(x, u) - M(y, v), \eta(x, y) \rangle \subseteq P;$$

- (b) η -pseudomonotone in C with respect to T , if for any $x, y \in X, u \in T(x)$, such that

$$\langle M(x, u), \eta(y, x) \rangle \subseteq Z \setminus (-\text{int}P(x))$$

implies that there exist $v \in T(y)$ such that

$$\langle M(y, v), \eta(y, x) \rangle \subseteq Z \setminus (-\text{int}P(x));$$

- (c) V -hemicontinuous if for any $x, y \in X, \alpha > 0$, and $u_\alpha \in T(x + \alpha y)$, there exist $u_0 \in T(x)$ such that for any $z \in K$,

$$\langle M(x, u_\alpha), z \rangle \rightarrow \langle M(x, u_0), z \rangle \text{ as } \alpha \rightarrow 0^+,$$

where $\langle M(x, u), z \rangle = \bigcup_{w \in M(x, u)} \{\langle w, z \rangle\}$.

Definition 2.4. A multivalued mapping $T : X \rightarrow 2^Y$ is said to be upper semi continuous on X if $\forall x \in X$, for each open set G in Y with $G \supset T(x)$, there exists an open neighbourhood $O(x)$ of x in X such that $G \supset T(x') \forall x' \in O(x)$.

3. Existence Results for GVVLP in Hausdörff Topological Vector Spaces

First, we prove the following theorem:

Theorem 3.1. Let X, Y and Z be three Hausdörff topological vector spaces, Z be regular and $K \subseteq X$ be a nonempty and compact convex subset and C be a nonempty subset of Y . Assume that

1. $P : K \rightarrow 2^Z$ is a multivalued mapping such that for each $x \in K$, $P(x)$ is closed, pointed, solid and convex cone with $\text{int}P(x) \neq \emptyset$;
2. $T : K \rightarrow 2^C$ is upper semicontinuous with compact valued mapping;
3. $\eta : K \times K \rightarrow X$ is continuous in the second argument and affine in the first argument. Also $\eta(x, x) = 0$ for all $x \in K$;
4. the multivalued mapping $W : K \rightarrow 2^Z$ defined by $W(x) = Z \setminus (-\text{int}P(x))$ is upper semicontinuous;
5. the multivalued mapping $M : K \times C \rightarrow 2^{L(X,Z)}$ is upper semicontinuous and affine in first argument;
6. $\forall x \in K, \exists u \in T(x)$ such that

$$\text{Max}\langle M(x, u), \eta(x, x) \rangle \subseteq Z \setminus (-\text{int}P(x)).$$

Then there exists $x_0 \in K$, such that $\forall y \in K, \exists u_0 \in T(x_0)$, such that

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subseteq Z \setminus (-\text{int}P(x_0)).$$

Proof. Consider

$$A := \{(y, x) \in K \times K : \exists u \in T(x), \text{ such that } \text{Max}\langle M(x, u), \eta(y, x) \rangle \subseteq Z \setminus (-\text{int}P(x))\}$$

Because of (6), $\forall y \in K, (y, y) \in A$. Now we claim that for each fixed $y \in K$, the set $A_y = \{x \in K : (y, x) \in A\}$ is closed. Indeed, let $\{x_\alpha\}$ be a net in A_y such that $x_\alpha \rightarrow x_0$. Since $x_\alpha \in A_y$, $\exists u_\alpha \in T(x_\alpha)$ such that

$$\text{Max}\langle M(x_\alpha, u_\alpha), \eta(y, x_\alpha) \rangle \subseteq W(x_\alpha) : Z \setminus (-\text{int}P(x_\alpha)).$$

By (2) it follows that $T(K) = \cup_{x \in K} T(x)$ is compact. Since $\{u_\alpha\} \subset T(K)$, $\{u_\alpha\}$ has a cluster point $u_0 \in T(K)$; without any loss of generality, we may assume

$u_\alpha \rightarrow u_0$, and hence upper semicontinuity gives $u_0 \in T(x_0)$. Since $\eta(y, \cdot)$, $\langle \cdot, \cdot \rangle$ are continuous, let $s_\alpha \in M(x_\alpha, u_\alpha)$ then by upper semicontinuity of M , we have $s_0 \in M(x_0, u_0)$, and hence

$$\langle s_\alpha, \eta(y, x_\alpha) \rangle \rightarrow \langle s_0, \eta(y, x_0) \rangle,$$

which implies that

$$\text{Max}\langle M(x_\alpha, u_\alpha), \eta(y, x_\alpha) \rangle \rightarrow \langle \text{Max}M(x_0, u_0), \eta(y, x_0) \rangle.$$

Since W is upper semicontinuous and Z is regular, we have $\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \in W(x_0)$ and hence $u_0 \in A_y$ implies that A_y is closed.

Next, we show that, for each fixed $x \in K$, $A_x = \{y \in K : (y, x) \notin A\}$ is convex. Let $y_1, y_2 \in A_x$ and $\lambda \in [0, 1]$; Since $\text{int}P(x)$ is convex, $\eta(\cdot, x)$ is affine, $\forall u \in T(x)$, we have

$$\text{Max}\langle M(x, u), \eta(y_1, x) \rangle \subseteq -\text{int}P(x)$$

and

$$\text{Max}\langle M(x, u), \eta(y_2, x) \rangle \subseteq -\text{int}P(x)$$

which imply that, $\forall u \in T(x)$,

$$\text{Max}\langle M(x, u), \eta(\lambda y_1 + (1 - \lambda)y_2, x) \rangle \subseteq -\text{int}P(x)$$

This gives that $\lambda y_1 + (1 - \lambda)y_2 \in A_x$ and thus A_x is convex.

By Theorem 2.2., $\exists x_0 \in K$ such that $K \times \{x_0\} \subset A$, i.e. $\forall y \in K$, $\exists u_0 \in T(x_0)$ such that

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subseteq Z \setminus (-\text{int}P(x_0))$$

and the proof is completed.

Definition 3.2. Let X be a topological space, I be a subset of X , such that $I = \bigcup_{n=1}^{\infty} I_n$ where $\{I_n\}_{n=1}^{\infty}$ is increasing (in the sense that $I_n \subseteq I_{n+1}$) sequence of non-empty compact sets. A sequence $\{x_n\}_{n=1}^{\infty}$ in I is said to be an *escaping sequence* from I (relative to $\{I_n\}_{n=1}^{\infty}$) iff for each $n = 1, 2, \dots$, $\exists m > 0$, such that $x_k \notin I_n$, $\forall k \geq m$.

In *non-compact* case, we prove the following result, using the concept of escaping sequence:

Theorem 3.2. Let X, Y and Z be three Hausdörff topological vector spaces; K be a subset of X such that $K = \bigcup_{n=1}^{\infty} K_n$ where $\{K_n\}_{n=1}^{\infty}$ is increasing sequence of non-empty and compact convex subsets of K ; Z be a regular and let C be a non-empty subset of Y . Let $N = \{1, 2, \dots\}$ and let the assumptions (1)-(6) of Theorem 3.1 hold. Further assume that:

1. for each sequence $\{x_n\}_{n=1}^{\infty}$ in K with $x_n \in K_n, \forall n = 1, 2, 3, \dots$, which is escaping from K relative to $\{K_n\}_{n=1}^{\infty}$, $\exists m \in N$ and $\exists z_m \in K_m$ such that, $\forall u_m \in T(x_m)$, we have

$$\text{Max}\langle M(x_m, u_m), \eta(z_m, x_m) \rangle \subseteq -\text{int}P(x)$$

Then, $\exists x_0 \in K$ such that, $\forall y \in K, \exists u_0 \in T(x_0)$ such that

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subseteq Z \setminus (-\text{int}P(x)).$$

Proof. Since by Theorem 3.1, $\forall n \in N, \exists x_n \in K_n$, such that $\forall z \in K_n, \exists u_n \in T(x_n)$ such that

$$\text{Max}\langle M(x_n, u_n), \eta(z, x_n) \rangle \subseteq Z \setminus (-\text{int}P(x_n)) \quad (3.1)$$

Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ be escaping from K relative $\{K_n\}_{n=1}^{\infty}$. By (1), $\exists m \in N$, and $\exists z_m \in K_m$, such that $\forall u \in T(x_m)$ we have

$$\text{Max}\langle M(x_m, u), \eta(z_m, x_m) \rangle \subseteq -\text{int}P(x_m)$$

which is a contradiction to (3.1). Therefore $\{x_n\}_{n=1}^{\infty}$ is not an escaping sequence from K relative to $\{K_n\}_{n=1}^{\infty}$; than $\exists p \in N$, and there is some subsequence $\{x_{j_n}\}$ of $\{x_n\}_{n=1}^{\infty}$ which must lie entirely in K_p . Since K_p is compact, there is a subsequence $\{x_{i_n}\}_{i_n \in \Gamma}$ of $\{x_{j_n}\}$ in K_p and $\exists y \in K_p$ such that $x_{i_n} \rightarrow y$ where $i_n \rightarrow \infty$. Since $\{K_n\}_{n=1}^{\infty}$ is increasing, $\forall x \in K, \exists i_0 \in \Gamma$ with $i_0 > p$, such that $x_0 \in K_{i_0}, \forall i_n \in \Gamma$ and $i_n > i_0$, we have $y \in K_{i_0} \subseteq K_{i_n}$, and $\exists u_{i_n} \in T(x_{i_n}) \subseteq T(K_p)$ such that

$$\text{Max}\langle M(x_{i_n}, u_{i_n}), \eta(y, x_{i_n}) \rangle \subseteq W(x_{i_n} := Z \setminus (-\text{int}P(x_{i_n}))). \quad (3.2)$$

By the same arguments used in the proof of Theorem 3.1, we can prove that (3.2) implies that

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subseteq Z \setminus (-\text{int}P(x_0)),$$

and then the proof is completed.

4. Existence Results for GVLIP in Banach Spaces

First, we prove the following result:

Theorem 4.1. Let X, Y and Z be Banach spaces. Let K be a nonempty closed subset of Y . Let the assumptions (1),(2),(3) of Theorem 3.1 hold. Further assume that

1. $\eta : K \times K \rightarrow X$ is continuous and affine in both arguments such that $\eta(x, x) = 0, \forall x \in K$;
2. $M : K \times C \rightarrow 2^{L(X,Z)}$ is η -pseudomonotone, V -hemicontinuous multivalued mapping;
3. there exists a nonempty and compact subset D of X such that for each finite subset L_N of K , such that $N \subset L_N$ and $\forall x \in L_N \setminus D, \exists u \in T(x)$ then there exists $y_1 \in L_N$, such that

$$\text{Max}\langle M(x, u), \eta(x, y_1) \rangle \cap (-\text{int}P(x)) \neq \emptyset.$$

Then there exists $x_0 \in K, \exists u_0 \in T(x_0)$ such that

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subseteq Z \setminus (-\text{int}P(x_0)), \text{ for all } y \in K.$$

Proof. Let $F : K \rightarrow 2^K$ and $G : K \rightarrow 2^K$ by

$$F(y) = \{x \in K : \exists u \in T(x) \text{ such that } \text{Max}\langle M(x, u), \eta(y, x) \rangle \subseteq Z \setminus (-\text{int}P(x))\}$$

$$G(y) = \{x \in K : \exists v \in T(y) \text{ such that } \text{Max}\langle M(y, v), \eta(y, x) \rangle \subseteq Z \setminus (-\text{int}P(x))\}$$

for each $y \in K$. Since T is upper semicontinuous with compact valued, so $M(.,.)$ is compact valued mapping, and $\langle ., . \rangle$ is continuous, $\langle M(x, u), \eta(y, x) \rangle$ and $\langle M(y, v), \eta(y, x) \rangle$ are both compact subset of Z . Evidently, we know that for each $y \in K, F(x)$ and $G(y)$ are both well defined. Since $\eta(.,.)$ is continuous and $\eta(x, x) = 0, \forall x \in K$. For each $y \in K, y \in F(y)$ and $y \in G(y)$, so that for each $y \in K, F(y) \neq \emptyset, G(y) \neq \emptyset$.

Next we prove that $F(y) \subset G(y), \forall y \in K$.

$$\text{Max}\langle M(x, u), \eta(y, x) \rangle \subseteq Z \setminus (-\text{int}P(x))$$

Since $\langle M(x, u), \eta(y, x) \rangle$ is a compact set in Z , then η -pseudomonotonicity of M .

$$\text{Max}\langle M(y, v), \eta(y, x) \rangle \subseteq Z \setminus (-\text{int}P(x)).$$

This implies that $x \in G(y)$. Therefore $F(y) \subset G(y)$, $\forall y \in K$.

Next we claim that F is a KKM-map. Indeed, let $\alpha_i \geq 0, 1 \leq i \leq n$, with $\sum_{i=1}^n \alpha_i = 1$. Suppose on the contrary that F is not a KKM-mapping. Then there exists a finite set $\{x_1, x_2, \dots, x_n\} \subset K$ such that

$$\text{con}\{x_1, x_2, \dots, x_n\} \not\subseteq \bigcup_{i=1}^n F(x_i).$$

Let $\alpha_i \geq 0, 1 \leq i \leq n$, with $\sum_{i=1}^n \alpha_i = 1$, therefore there exists $x = \sum_{i=1}^n \alpha_i x_i \in \text{con}\{x_1, x_2, \dots, x_n\}$, $\Rightarrow x \notin \bigcup_{i=1}^n F(x_i)$, then for any $u \in T(x)$,

$$\text{Max}\langle M(x, u), \eta(x_i, x) \rangle \not\subseteq Z \setminus (-\text{int}P(x)), \quad i = 1, 2, \dots, n.$$

Since $\eta(., .)$ is affine, we have, there exist $w_i \in M(x, u)$, $i = 1, 2, \dots, n$, such that

$$\langle w_i, \eta(x_i, x) \rangle \in -\text{int}P(x), \quad i = 1, 2, \dots, n.$$

Since $\text{int}P(x)$ is convex, we have

$$\sum_{i=1}^n \alpha_i \langle w_i, \eta(x_i, x) \rangle \in -\text{int}P(x).$$

Again, since

$$\begin{aligned} \sum_{i=1}^n \alpha_i \langle w_i, \eta(x_i, x) \rangle &\in \sum_{i=1}^n \alpha_i \langle M(x, u), \eta(x_i, x) \rangle \\ &= \langle M(x, u), \eta(\sum_{i=1}^n \alpha_i x_i, x) \rangle && [\text{by condition(1)}] \\ &= \langle M(x, u), \eta(x, x) \rangle = 0, \end{aligned}$$

and thus $\sum_{i=1}^n \alpha_i \langle w_i, \eta(x_i, x) \rangle = 0 \in -\text{int}P(x)$ which is a contradict to the assumption that $P(x)$ is a pointed convex cone. Hence F is a KKM mapping and thus G is a KKM-mapping.

Now we claim that $G(y)$ is closed for each $y \in K$. Indeed, let $\{x_n\} \subset G(y)$ be a sequence such that $x_n \rightarrow x \in K$. Since $x_n \in G(y)$, $\forall n$, $\exists v_n \in T(y)$ such that

$$\text{Max}\langle M(y, v_n), \eta(y, x_n) \rangle \subset W(x_n) := Z \setminus (-\text{int}P(x_n)), \quad i = 1, 2, 3, \dots$$

Since $T(y)$ is compact then without loss of generality, we can assume that, $\exists v_0 \in T(y)$, such that $v_n \rightarrow v_0$. Since $\langle \cdot, \cdot \rangle$ is continuous and W is upper semicontinuous, we have

$$\langle M(y, v_n), \eta(y, x_n) \rangle \rightarrow \langle M(y, v_0), \eta(y, x) \rangle \subseteq W(x)$$

implies that $x \in G(y)$ and hence $G(y)$ is closed.

Now, we prove that

$$\bigcap_{y \in K} G(y) \neq \emptyset.$$

In fact, by condition (3) for all finite set $N \subset K$, there exists a nonempty compact convex subset $L_N \subset K$, such that $N \subset L_N$ and $\forall x \in L_N \setminus D$, $\exists y \in L_N$ such that $x \notin G(y)$. Hence we have

$$L_N \cap \{G(y) : y \in L_N\} \subset D.$$

By Park [11, p.27], it follows that

$$\bigcap_{y \in K} G(y) \neq \emptyset. \quad (4.1)$$

Finally, we prove that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y). \quad (4.2)$$

Since $F(y) \subset G(y)$ for all $y \in K$ we have

$$\bigcap_{y \in K} F(y) \subset \bigcap_{y \in K} G(y).$$

Now, it is sufficient to prove

$$\bigcap_{y \in K} G(y) \subseteq \bigcap_{y \in K} F(y)$$

In fact, if $x \in \bigcap_{y \in K} G(y)$, then for each $y \in K$, $x \in G(y)$, $\exists v \in T(y)$ such that

$$\text{Max}\langle M(y, v), \eta(y, x) \rangle \subset Z \setminus (-\text{int}P(x)).$$

Since K is convex, for any $\alpha \in (0, 1)$, $y_\alpha = \alpha y + (1 - \alpha)x \in K$. Hence there exists $v_\alpha \in T(y_\alpha)$ such that

$$\text{Max}\langle M(y_\alpha, v_\alpha), \eta(y_\alpha, x) \rangle \subset Z \setminus (-\text{int}P(x))$$

or

$$\begin{aligned} \text{Max}\langle M(y_\alpha, v_\alpha), \eta(y_\alpha, x) \rangle &= \text{Max}\langle M(y_\alpha, v_\alpha), \eta(\alpha y + (1 - \alpha)x, x) \rangle \\ &= \text{Max}\langle M(y_\alpha, v_\alpha), \alpha \eta(y, x) \rangle \quad [\text{by condition(1)}] \\ &\subseteq Z \setminus (-\text{int} P(x)). \end{aligned}$$

This implies that for each $\alpha \in (0, 1)$.

$$\text{Max}\langle M(y_\alpha, v_\alpha), \alpha \eta(y, x) \rangle \bigcap (-\text{int} P(x)) = \emptyset$$

Dividing by α , we have

$$\text{Max}\langle M(y_\alpha, v_\alpha), \eta(y, x) \rangle \bigcap (-\text{int} P(x)) = \emptyset.$$

Letting $\alpha \rightarrow 0^+$ and using the V-hemicontinuity of M , there exists $v_0 \in T(x)$, such that

$$\lim_{\alpha \rightarrow 0^+} \text{Max}\langle M(y_\alpha, v_\alpha), \eta(y, x) \rangle = \text{Max}\langle M(x, v_0), \eta(y, x) \rangle \subseteq Z \setminus (-\text{int} P(x))$$

i.e, $x \in F(y)$ for all $y \in K$. Therefore $x \in \bigcap_{y \in K} F(y)$, and so that $\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y)$ is true.

Combining the cases of (4.1) and (4.2) above, we know that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset.$$

Taking $x_0 \in \bigcap_{y \in K} F(y)$, $\exists u_0 \in T(x_0)$ such that

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subseteq Z \setminus (-\text{int} P(x_0)), \quad \forall y \in K.$$

This completes the proof of the theorem.

Finally, we prove the following result:

Theorem 4.2. Let X, Y and Z be Banach spaces. Let K be a non-empty closed and convex subset of X and let C be a non-empty subset of Y . Let the assumptions (1),(2),(4) of Theorem 3.1 hold. Further, we assume that:

1. $\eta : K \times K \rightarrow X$ is continuous and affine such that $\eta(x, x) = 0, \quad \forall x \in K$;
2. $M : K \times C \rightarrow 2^{L(X, Z)}$ is η - pseudomonotone multivalued mapping;

3. for each $x \in K$, $\exists x_0 \in K$ such that $u_0 \in T(x_0)$ and

$$\text{Max}\langle M(x_0, u_0), \eta(x_0, x) \rangle \subseteq Z \setminus (-\text{int}P(x))$$

4. $\exists$ a nonempty set K_0 contained in a compact and convex subset K_1 of K such that

$$D = \bigcap_{x_0 \in K_0} \bigcap_{u_0 \in T(x_0)} \{x \in K : \text{Max}\langle M(x_0, u_0), \eta(x_0, x) \rangle \subseteq W(x)\}$$

is either empty or compact.

Then there exist $x_0 \in K$, $u_0 \in T(x_0)$ such that

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subseteq Z \setminus (-\text{int}P(x_0)).$$

Proof. We establish the proof by an indirect method of showing a contradiction. Suppose that there does not exist $x_0 \in K$ and $u_0 \in T(x_0)$ such that

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \not\subseteq Z \setminus (-\text{int}P(x_0))$$

i.e.,

$$\text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subset -\text{int}P(x_0).$$

Since $x_0 \in K$, then the set

$$F(x_0) = \{y \in K : \exists u_0 \in T(x_0) \text{ such that } \text{Max}\langle M(x_0, u_0), \eta(y, x_0) \rangle \subset -\text{int}P(x_0)\}$$

is nonempty. Since $\eta(., .)$ is affine then we can easily show that the set $F(x_0)$ is convex for each $x_0 \in K$.

Thus $F : K \rightarrow 2^{L(X, Z)}$ is multivalued mapping with $F(x_0)$ nonempty and convex for each $x_0 \in K$.

Now, for each $x_0 \in K$

$$\begin{aligned} F^{-1}(x_0) &= \{x \in K : x_0 \in F(x)\} \\ &= \{x \in K : \exists u \in T(x) \text{ such that } \text{Max}\langle M(x, u), \eta(x_0, x) \rangle \subset \text{int}P(x)\}. \end{aligned}$$

Using η -pseudomonotonicity of M , for each $x_0 \in K$, the complement of $F^{-1}(x_0)$ is in K , that is

$$[F^{-1}(x_0)]^c = \{x \in K : \exists u \in T(x), \text{Max}\langle M(x, u), \eta(x_0, x) \rangle \subset W(x)\}$$

$$\begin{aligned} &\subseteq \{x \in K : \exists u_0 \in T(x_0), \text{Max}\langle M(x_0, u_0), \eta(x_0, x) \rangle \subset W(x)\} \\ &= B(x_0) \subseteq K. \end{aligned}$$

Let $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and $x_1, x_2 \in B(x_0)$, $\exists u_0 \in T(x_0)$ such that

$$\alpha \text{Max}\langle M(x_0, u_0), \eta(x_0, x_1) \rangle \subseteq \alpha W(x_1),$$

$$\beta \text{Max}\langle M(x_0, u_0), \eta(x_0, x_2) \rangle \subseteq \beta W(x_2).$$

Since η is affine and W is convex, we have

$$\begin{aligned} \text{Max}\langle M(x_0, u_0), \eta(x_0, \alpha x_1 + \beta x_2) \rangle &\subseteq \alpha W(x_1) + \beta W(x_2) \\ &\subseteq W(\alpha x_1 + \beta x_2). \end{aligned}$$

This implies that $B(x_0)$ is convex.

Also upper semicontinuity of compact valued mapping T , upper semicontinuity of W , continuity of $\eta(.,.)$ and $\langle ., . \rangle$ yield the relatively closeness of $B(x_0)$.

Hence, for each $x_0 \in K$, $O_{x_0} = [B(x_0)]^c$ is a relatively open subset of K . Now, by assumption (3), it follows that $\bigcup_{x_0 \in K} O_{x_0} = K$. Indeed if $x \in K$ by assumption (3), $\exists x_0 \in K$, $u_0 \in T(x_0)$ such that $x \in [B(x_0)]^c = O_{x_0}$. Thus $x \in \bigcup_{x_0 \in K} O_{x_0}$. Finally

$$D = \bigcap_{x_0 \in K} \bigcap_{u_0 \in T(x_0)} B(x_0) = \bigcap_{x_0 \in K} \bigcap_{u_0 \in T(x_0)} O_{x_0}^c$$

is compact or empty by assumption (4). Hence, from Theorem 2.1, $\exists x_0 \in K$, such that $x_0 \in F(x_0)$, that is, $0 \in -\text{int}P(x_0)$, which is impossible. The proof is completed.

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