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ECCENTRIC AND SUPER ECCENTRIC SIGNED GRAPHS

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Abstract: In this paper we introduced the new notions eccentric and super eccentric signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterizations of these notions. Further, we presented some switching equivalent characterizations.

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1. Introduction

For standard terminology and notation in graph theory we refer Harary [4] and Zaslavsky [14] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

In a graph Γ, the distance $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(\Gamma)$ of Γ is defined by $r(\Gamma) = \min\{e(u) : u \in \Gamma\}$ and the diameter $d(\Gamma)$ of Γ is defined by $d(\Gamma) = \max\{e(u) : u \in \Gamma\}$. A graph for which $r(\Gamma) = d(\Gamma)$ is called a *self-centered graph* of radius $r(\Gamma)$.

Let $\Gamma = (V, E)$ be a simple undirected graph. The eccentricity $e(v)$ of a vertex in $V(\Gamma)$ is defined by $e(v) = \max_{u \in V} d(u,v)$, where $d(u,v)$ stands for the length of the shortest path in Γ between u and v. In case Γ is disconnected and u and v belong to different components, we set $d(u,v) = +\infty$.

Akiyama et al. [2] defined the eccentric graph $\mathcal{E}(\Gamma)$ of Γ as a graph on the same set of vertices as Γ obtained, by joining two vertices if and only if $d(u,v) = \min\{e(u), e(v)\}$.

Iqbalunnisa et al. [8] defined the super eccentric graph $\mathcal{SE}(\Gamma)$ of a graph Γ on the same set of vertices as Γ where the adjacency relation between vertices is defined by $d(u,v) \geq \mathrm{rad}(\Gamma)$ while Γ is connected and when Γ is disconnected, two vertices are adjacent in $\mathcal{SE}(\Gamma)$ if they belong to different components of Γ.

A signed graph $\Sigma = (\Gamma, \sigma) = (V, E, \sigma)$ is a graph Γ together with a function that assigns a sign $\sigma(e) \in \{+, -\}$, to each edge in Γ. σ is called the signature or sign function. In such a signed graph, a subset A of $E(\Gamma)$ is said to be positive if it contains an even number of negative edges, otherwise is said to be negative. Balance or imbalance is the fundamental property of a signed graph. A signed graph Σ is balanced if each cycle of Σ is positive. Otherwise it is unbalanced.

Signed graphs Σ_1 and Σ_2 are isomorphic, written $\Sigma_1 \cong \Sigma_2$, if there is an isomorphism between their underlying graphs that preserves the signs of edges.

The theory of balance goes back to Heider [7] who asserted that a social system is balanced if there is no tension and that unbalanced social structures exhibit a tension resulting in a tendency to change in the direction of balance. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists. In 1956, Cartwright and Harary [3] provided a mathematical model for balance through graphs.

A *marking* of Σ is a function $\zeta : V(\Gamma) \to \{+, -\}$. Given a signed graph Σ one can easily define a marking ζ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\zeta(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking ζ of Σ is called *canonical marking* of Σ.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1. *A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [5]).*

(ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. (Sampathkumar [10]).*

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. *Complement* of Σ is a signed graph $\overline{\Sigma} = (\overline{\Gamma}, \sigma')$, where for any edge $e = uv \in \overline{\Gamma}$, $\sigma'(uv) = \zeta(u)\zeta(v)$. Clearly, $\overline{\Sigma}$ as defined here is a balanced signed graph due to Theorem 1.

A switching function for Σ is a function $\zeta : V \to \{+, -\}$. The switched signature is $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has end points v, w. The switched signed graph is $\Sigma^\zeta := (\Sigma|\sigma^\zeta)$. We say that Σ switched by ζ. Note that $\Sigma^\zeta = \Sigma^{-\zeta}$ (see [1]).

If $X \subseteq V$, switching Σ by X (or simply switching X) means reversing the sign of every edge in the cutset $E(X, X^c)$. The switched signed graph is Σ^X. This is the same as Σ^ζ where $\zeta(v) := -$ if and only if $v \in X$. Switching by ζ or X is the same operation with different notation. Note that $\Sigma^X = \Sigma^{X^c}$.

Signed graphs Σ_1 and Σ_2 are switching equivalent, written $\Sigma_1 \sim \Sigma_2$ if they have the same underlying graph and there exists a switching function ζ such that $\Sigma_1^\zeta \cong \Sigma_2$. The equivalence class of Σ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\},$$

is called the its switching class.

Similarly, Σ_1 and Σ_2 are switching isomorphic, written $\Sigma_1 \cong \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2. The equivalence class of Σ is called its switching isomorphism class.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [12]) or *cycle isomorphic* (see [13]) if there exists an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known (see [13]):

Theorem 2. *(**T. Zaslavsky** [13]) Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In [11], the authors introduced the switching and cycle isomorphism for signed digraphs.

In this paper, we initiate a study of the *eccentric signed graph and super eccentric signed graph* of a given signed graph and solve some important signed graph equations and equivalences involving it. Further, we obtained the structural characterization of these notions.

2. Eccentric Signed Graph of a Signed Graph

Motivated by the existing definition of complement of a signed graph, we now extend the notion of eccentric graphs to signed graphs as follows: The *eccentric* *signed graph* $\mathcal{E}(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ is a signed graph whose underlying graph is $\mathcal{E}(\Gamma)$ and sign of any edge uv is $\mathcal{E}(\Sigma)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of Σ. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called eccentric signed graph, if $\Sigma \cong \mathcal{E}(\Sigma')$ for some signed graph Σ'. The following result restricts the class of eccentric graphs.

Theorem 3. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its eccentric signed graph $\mathcal{E}(\Sigma)$ is balanced.*

Proof. Since sign of any edge $e = uv$ in $\mathcal{E}(\Sigma)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of Σ, by Theorem 1, $\mathcal{E}(\Sigma)$ is balanced.

For any positive integer k, the k^{th} iterated eccentric signed graph, $\mathcal{E}^k(\Sigma)$ of Σ is defined as follows:

$$\mathcal{E}^0(\Sigma) = \Sigma,\ \mathcal{E}^k(\Sigma) = \mathcal{E}(\mathcal{E}^{k-1}(\Sigma)).$$

Corollary 4. *For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k, $\mathcal{E}^k(\Sigma)$ is balanced.*
The following result characterize signed graphs which are eccentric signed graphs.

Theorem 5. *A signed graph $\Sigma = (\Gamma, \sigma)$ is an eccentric signed graph if, and only if, Σ is balanced signed graph and its underlying graph Γ is an eccentric graph.*

Proof. Suppose that Σ is balanced and Γ is an eccentric graph. Then there exists a graph Γ' such that $\mathcal{E}(\Gamma') \cong \Gamma$. Since Σ is balanced, by Theorem 1, there exists a marking ζ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $\mathcal{E}(\Sigma') \cong \Sigma$. Hence Σ is an eccentric signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is an eccentric signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $\mathcal{E}(\Sigma') \cong \Sigma$. Hence, Γ is the eccentric graph of Γ' and by Theorem 3, Σ is balanced.

Let $S_i = \{v \in V(\Gamma) | e(v) = i\}$, $i = 1, 2, \cdots$. In [2], the authors completely characterize those graphs whose eccentric graph is isomorphic to its complement.

Theorem 6. *$\mathcal{E}(\Gamma) \cong \overline{\Gamma}$ if and only if $S_i = \phi$, $i = 1, 4, 5, 6, \cdots$ and no two vertices in S_3 have a common nieghbour.*

In view of the above result, we have the following result that characterizes the family of signed graphs satisfies $\mathcal{E}(\Sigma) \sim \overline{\Sigma}$.

Theorem 7. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\mathcal{E}(\Sigma) \sim \overline{\Sigma}$ if, and only if, Γ is a graph with $S_i = \phi$, $i = 1, 4, 5, 6, \cdots$ and no two vertices in S_3 have a common nieghbour.*

Proof. Suppose that $\mathcal{E}(\Sigma) \sim \overline{\Sigma}$. Then clearly, $\mathcal{E}(\Gamma) \cong \overline{\Gamma}$. Hence by Theorem 6, Γ is a graph with $S_i = \phi$, $i = 1, 4, 5, 6, \cdots$ and no two vertices in S_3 have a common nieghbour.

Conversely, suppose that Σ is a signed graph whose underlying graph Γ is a graph $S_i = \phi$, $i = 1, 4, 5, 6, \cdots$ and no two vertices in S_3 have a common nieghbour. Then by Theorem 6, $\mathcal{E}(\Gamma) \cong \overline{\Gamma}$. Since for any signed graph Σ, both $\mathcal{E}(\Sigma)$ and $\overline{\Sigma}$ are balanced, the result follows by Theorem 2.

The following result characterizes the signed graphs which are isomorphic to eccentric signed graphs. In case of graphs the following result is due to Akiyama et al. [8]:

Theorem 8. *If $r(\Gamma) = 1$, then $\mathcal{E}(\Gamma) \cong \Gamma$ if and only if $< V - S_1 >_\Gamma$ is self-complementary, where S_1 denotes the set of vertices in Γ of eccentricity 1.*

Theorem 9. *A signed graph $\Sigma = (\Gamma, \sigma)$ with $r(\Gamma) = 1$, $\Sigma \sim \mathcal{E}(\Sigma)$ if, and only if, Σ is balanced and $< V - S_1 >_\Gamma$ is self-complementary, where S_1 denotes the set of vertices in Γ of eccentricity 1.*

Proof. Suppose $\mathcal{E}(\Sigma) \sim \Sigma$. This implies, $\mathcal{E}(\Gamma) \cong \Gamma$ and hence by Theorem 8, we see that the graph Γ satisfies the conditions in Theorem 8. Now, if Σ is any signed graph with $< V - S_1 >_\Gamma$ is self-complementary, where S_1 denotes the set of vertices in Γ of eccentricity 1, Theorem 3 implies that $\mathcal{S}(\Sigma)$ is balanced and hence if Σ is unbalanced and its eccentric signed graph $\mathcal{E}(\Sigma)$ being balanced can not be switching equivalent to Σ in accordance with Theorem 2. Therefore, Σ must be balanced.

Conversely, suppose that Σ balanced signed graph with $< V - S_1 >_\Gamma$ is self-complementary, where S_1 denotes the set of vertices in Γ of eccentricity 1. Then, since $\mathcal{E}(\Sigma)$ is balanced as per Theorem 3 and since $\mathcal{E}(\Gamma) \cong \Gamma$ by Theorem 8, the result follows from Theorem 2 again.

3. Super Eccentric Signed Graph of a Signed Graph

Motivated by the existing definition of complement of a signed graph, we now extend the notion of super eccentric graphs to signed graphs as follows: The *super eccentric signed graph* $\mathcal{SE}(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ is a signed graph whose underlying graph is $\mathcal{SE}(\Gamma)$ and sign of any edge uv is $\mathcal{SE}(\Sigma)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of Σ. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called super eccentric signed graph, if $\Sigma \cong \mathcal{SE}(\Sigma')$ for some signed graph Σ'. The following result restricts the class of eccentric graphs.

Theorem 10. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its super eccentric signed graph $\mathcal{SE}(\Sigma)$ is balanced.*

Proof. Since sign of any edge $e = uv$ in $\mathcal{SE}(\Sigma)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of Σ, by Theorem 1, $\mathcal{SE}(\Sigma)$ is balanced.

For any positive integer k, the k^{th} iterated super eccentric signed graph, $\mathcal{SE}^k(\Sigma)$ of Σ is defined as follows:

$$\mathcal{SE}^0(\Sigma) = \Sigma,\ \mathcal{SE}^k(\Sigma) = \mathcal{SE}(\mathcal{SE}^{k-1}(\Sigma)).$$

Corollary 11. For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k, $\mathcal{SE}^k(\Sigma)$ is balanced.

The following result characterize signed graphs which are super eccentric signed graphs.

Theorem 12. *A signed graph $\Sigma = (\Gamma, \sigma)$ is a super eccentric signed graph if, and only if, Σ is balanced signed graph and its underlying graph Γ is a super eccentric graph.*

Proof. Suppose that Σ is balanced and Γ is a super eccentric graph. Then there exists a graph Γ' such that $\mathcal{SE}(\Gamma') \cong \Gamma$. Since Σ is balanced, by Theorem 1, there exists a marking ζ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $\mathcal{SE}(\Sigma') \cong \Sigma$. Hence Σ is a super eccentric signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is a super eccentric signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $\mathcal{SE}(\Sigma') \cong \Sigma$. Hence, Γ is the eccentric graph of Γ' and by Theorem 10, Σ is balanced.

In [9], the author characterize those graphs whose super eccentric graph is isomorphic to its complement.

Theorem 13. *For any graph Γ, $\mathcal{SE}(\Gamma) \cong \overline{\Gamma}$ if and only if $r(\Gamma) = 2$ or Γ is disconnected with each component complete.*

In view of the above result, we have the following result that characterizes the family of signed graphs satisfies $\mathcal{SE}(\Sigma) \sim \overline{\Sigma}$.

Theorem 14. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\mathcal{SE}(\Sigma) \sim \overline{\Sigma}$ if, and only if, Γ is a graph with $r(\Gamma) = 2$ or Γ is disconnected with each component complete.*

Proof. Suppose that $\mathcal{SE}(\Sigma) \sim \overline{\Sigma}$. Then clearly, $\mathcal{SE}(\Gamma) \cong \overline{\Gamma}$. Hence by Theorem 13, Γ is a graph with $r(\Gamma) = 2$ or Γ is disconnected with each component complete.

Conversely, suppose that Σ is a signed graph whose underlying graph Γ is a graph with $r(\Gamma) = 2$ or Γ is disconnected with each component complete. Then by Theorem 13, $\mathcal{SE}(\Gamma) \cong \overline{\Gamma}$. Since for any signed graph Σ, both $\mathcal{SE}(\Sigma)$ and $\overline{\Sigma}$ are balanced, the result follows by Theorem 2.

4. Negation of $\mathcal{E}(\Sigma)$ and $\mathcal{SE}(\Sigma)$

The notion of *negation* $\eta(\Sigma)$ of a given signed graph Σ defined in [6] as follows: $\eta(\Sigma)$ has the same underlying graph as that of Σ with the sign of each edge opposite to that given to it in Σ. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(.)$ of taking the negation of Σ.

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $\mathcal{E}(\Sigma)$ and $\mathcal{SE}(\Sigma)$ are balanced. We now examine, the conditions under which negation $\eta(\Sigma)$ of $\mathcal{E}(\Sigma)$ and $\mathcal{SE}(\Sigma)$ are balanced.

Theorem 15. *Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. If $\mathcal{E}(\Gamma)$ ($\mathcal{ES}(\Gamma)$) is bipartite then $\eta(\mathcal{E}(\Sigma))$ ($\eta(\mathcal{SE}(\Sigma))$) is balanced.*

Proof. Since, $\mathcal{E}(\Sigma)$ ($\mathcal{SE}(\Sigma)$) is balanced, if each cycle C in $\mathcal{E}(\Sigma)$ ($\mathcal{SE}(\Sigma)$) contains even number of negative edges. Also, since $\mathcal{E}(\Gamma)$ ($\mathcal{ES}(\Gamma)$) is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{E}(\Sigma)$ ($\mathcal{SE}(\Sigma)$) is also even. Hence $\eta(\mathcal{E}(\Sigma))$ ($\eta(\mathcal{SE}(\Sigma))$) is balanced.

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