South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752**
Vol.14, No.1 2018, pp. 59-64

PROPERTIES OF CERTAIN CLASS OF ANALYTIC FUNCTIONS

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(Received: January 08, 2018)

Abstract: We define here a simple class of negative coefficient. Properties like closure, linear combination and inclusion relations are investigated.

Keywords and Phrases: Analytic function, convex linear combination, Hadamard product.

2010 Mathematics Subject Classification: 30C45.

1. Introduction and Definitions

Let $\mathcal{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

in the open unit disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in U.$$

This class is denoted by $S^*(\alpha)$. In a similar vein the class of convex functions of order α is given by

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in U.$$

The class of positive real parts and bounded turning are familiar subclasses of analytic functions that have attracted the attention of several researchers in this area of Geometric function theory, see [5, 9, 10]. Tuneski et al [8] studied the linear combination of bounded turning and that of positive real part. So many other authors have considered the study of the above classes, some considered geometric combinations, where the combination in question is a special case, or of similar kind. See [4]. Other related works on negative coefficient can be found in [1,7,11, 12]. Motivated by the class introduced and studied in [8], and the methods of investigations by [1,2,3,4,6,11,13], we define the class $B_\alpha(\beta)$ as follows:

Let

$$P_\alpha(\beta) = \{f \in S : \Re[\,\alpha f'(z) + (1-\alpha)\frac{f(z)}{z}\,] > \beta, z \in U\}\,, \tag{1.2}$$

also define

$$B_\alpha(\beta) = \{f \in P_\alpha(\beta) \cap D_\alpha : \Re[\,\alpha f'(z) + (1-\alpha)\frac{f(z)}{z}\,] > \beta, z \in U\}\,, \tag{1.3}$$

where D_α denote the subclass of S consisting of functions f of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k. \tag{1.4}$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are anlytic in U, then their Hadamard product $f * g$ is given by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{1.5}$$

Note that the convolution so defined is also analytic in U.

Our main focus on this paper is to characterize the class $B_\alpha(\beta)$, with special emphasis, on coefficients bounds, extreme points and inclusion relations.

2. Coefficient Inequality

Theorem 1. Let the function $f(z)$ be defined as in (1.4), then $f \in B_\alpha(\beta)$ iff

$$\sum_{k=2}^{\infty} [\alpha(k-1)+1]\, a_k \leq 1-\beta \tag{2.1}$$

Proof. Assume that the inequality in (2.1)holds then for $|z| = 1$

$$\left|\alpha f'(z) + (1-\alpha)\frac{f(z)}{z} - 1\right|$$

$$= \left| \alpha \left(1 - \sum_{k=2}^{\infty} k a_k z^{k-1} \right) + (1-\alpha) - \sum_{k=2}^{\infty} (1-\alpha) a_k z^{k-1} - 1 \right|$$

$$= \left| \alpha - \sum_{k=2}^{\infty} \alpha k a_k z^{k-1} + (1-\alpha) - \sum_{k=2}^{\infty} (1-\alpha) a_k z^{k-1} - 1 \right|$$

$$\leq \left| \sum_{k=2}^{\infty} (\alpha(k-1)+1) a_k || z^{k+1} \right| \leq 1 - \beta$$

$$\leq \left| \sum_{k=2}^{\infty} (\alpha(k-1)11) a_k \right| \leq 1 - \beta$$

Conversely, suppose that $f(z)$ defined by (1.4)is in $B_\alpha(\beta)$, then it implies that

$$\Re \left\{ \alpha f'(z) + (1-\alpha) \frac{f(z)}{z} \right\} > \beta.$$

So that

$$\Re \left\{ \alpha \left(1 - \sum_{k=2}^{\infty} k a_k z^{k-1} \right) + (1-\alpha) \left(1 - \sum_{k=2}^{\infty} a_k z^{k-1} \right), \right\} > \beta. \qquad (2.2)$$

Having taken z on the real axis, such that $z \to 1^{-1}$ through the real axis, (2.1) is real , thus we have

$$- \left(\sum_{k=2}^{\infty} [\alpha(k-1)+1] a_k \right) \geq \beta - 1,$$

and therefore we have

$$\sum_{k=2}^{\infty} [\alpha(k-1)+1] a_k \leq 1 - \beta.$$

This proves our result. The result is best possible for the extremal function

$$f(z) = z - \frac{1-\beta}{[\alpha(k+1)+1]} z^k, k \geq 2. \qquad (2.3)$$

Corollary 1. Let the function given by (1.4) be in $B_\alpha(\beta)$, then

$$|a_k| \leq \frac{1-\beta}{[\alpha(k-1)+1]}, k \geq 2. \qquad (2.4)$$

If $\alpha = 0$ in Corollary 1, we get

Corollary 2. Let the function f given by (1.4) be in $B_\alpha(\beta)$, then

$$|a_k| \leq (1-\beta), k \geq 2. \tag{2.5}$$

If $\alpha = 1$ in Corollary 1, then we get

Corollary 3. Let the function f given by (1.4) be in $B_\alpha(\beta)$, then

$$|a_k| \leq \frac{1-\beta}{[(k-1)+1]}, k \geq 2. \tag{2.6}$$

Equality is attain by the function given by (2.3).

We shall further characterize the class by the following:

Theorem 2. Let $0 \leq \beta < 1, 0 \leq \alpha_1 \leq \alpha_2, \quad (\alpha_1, \alpha_1) \in N_\circ$, then $B_{\alpha_1}(\beta) \subseteq B_{\alpha_2}(\beta)$.

Proof. Considering (2.1), it is obvious to see that

$$\sum_{k=2}^{\infty} [\alpha_1(k-1)+1]a_k \leq \sum_{k=2}^{\infty} [\alpha_2(k-1)+1]a_k \leq 1-\beta. \tag{2.7}$$

Expression (2.7) is valid for all values of k for $f \in B_\alpha(\beta)$, hence it follows that $B_{\alpha_1}(\beta) \subseteq B_{\alpha_2}(\beta)$. This concludes our proof.

Theorem 3. Let $0 \leq \alpha < 1$ and $(\alpha_1, \alpha_1) \in N_\circ$, then, $B_{\alpha+1}(\beta) \subseteq B_\alpha(\beta)$.

Proof. The inclusion relation is obvious from Theorem 2.

3. Closure Property

Let $f_j(z)$ be defined for $j = 1, 2, 3, \cdots, m$ by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k, \quad (a_{k,j}) \geq 0 \tag{3.1}$$

Theorem 4. Let the function defined by (3.1) be in the class $B_\alpha(\beta)$ for every $j = 1, 2, 3, \cdots, m$.Then the function $h(z)$ define by

$$h(z) = \sum_{k=2}^{\infty} c_j f_j(z), \ (c_j) \geq 0, \tag{3.2}$$

is also in $B_\alpha(\beta)$ where $\sum c_j = 1$.

Proof. By (3.2), $h(z)$ can be written in the form

$$h(z) = z - \sum_{k=2}^{\infty} \Big(\sum c_j a_{k,j} \Big). \tag{3.3}$$

By definition of $f(z)$ and the condition of $f \in B_\alpha(\beta)$ we have

$$\sum_{k=2}^{\infty} [\alpha(k-1)+1]\, a_{k,j} \leq 1-\beta,$$

for every $j = 1, 2, 3, \cdots, m$ this implies that

$$\sum_{k=2}^{\infty} [\alpha(k-1)+1] \left(\sum_{j=1}^{m} c_j a_{k,j} \right)$$

$$= \sum_{j=1}^{m} c_j \left(\sum_{k=2}^{\infty} [\alpha(k-1)+1]\, a_{k,j} \right)$$

$$\leq \left(\sum_{j=1}^{m} c_j \right) (1-\beta)\,,$$

by (3.3) we have

$$\leq \left(\sum_{j=1}^{m} c_j \right) (1-\beta) = 1-\beta,$$

which by implication shows that $h(z)$ belongs to $B_\alpha(\beta)$.

Conflict of interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgement

The authors would like to acknowledge and appreciate the support received from Universiti Kebangsaan Malaysia and under the grant: UKM-2017-064.

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