J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.1, No.2 (2012), pp. 61-70

Certain results involving Eta-function

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(Received November 07, 2012)

Abstract: In this paper, making use of a result due to Denis, Singh and Singh [2], we have established certain results involving Eta-functions.
Keywords: Basic hypergeometric series, basic bilateral hypergeometric series, Eta-function, summation formula, transformation formula.
AMS subject classification code: Primary 33D90, 11A55 ; Secondary 11F20.

1. Introduction, Notations and Definitions:

In this paper we shall establish certain results involving eta-functions. This paper consists of three parts which include results involving eta-functions.
Eta function is defined as,

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) = q^{\frac{1}{24}} [q; q]_\infty, \tag{1.1}$$

where $q = e^{2\pi iz}$.

Part - I

We shall be in need of the following known results,

$$ {}_2\Psi_2 \left[\begin{array}{l} aq_1^m, yq_1^m; q_1; xq_1 \\ q_1^{1+m}, ayq_1^{1+m} \end{array} \right] {}_2\Phi_1 \left[\begin{array}{l} \alpha q, \beta q; q; x \\ \alpha\beta q \end{array} \right] $$

$$ = \frac{[\alpha, \beta; q]_m [q_1, ayq_1; q_1]_m}{[q, \alpha\beta q; q]_m [a, y; q_1]_m} \left(\frac{q}{q_1} \right)^m $$

$$ \times {}_2\Psi_2 \left[\begin{array}{l} \alpha q^m, \beta q^m; q; xq \\ q^{1+m}, \alpha\beta q^{1+m} \end{array} \right] {}_2\Phi_1 \left[\begin{array}{l} aq_1, yq_1; q_1; x \\ ayq_1 \end{array} \right], \tag{1.2}$$

[Denis, Singh and Singh 2; (4.4)]

where max $(|q|, |q_1|) < |x| < 1$.

Ramanujan's sum

$$ {}_1\Psi_1\left[\begin{array}{l} a;q;z \\ b \end{array}\right] = \frac{\left[q,\dfrac{b}{c},az,\dfrac{q}{az};q\right]_\infty}{\left[b,\dfrac{q}{a},z,\dfrac{b}{az};q\right]_\infty} \tag{1.3} $$

[Gasper and Rahman 3; App.II(29)]

2. Main Results

In this section we shall establish our main results.

Taking $a = q_1, \alpha = q$ in (1.2) and then summing the ${}_1\Psi_1$-series of both sides by making use of (1.3), we get

$$ \frac{{}_2\Phi_1\left[\begin{array}{l} q_1^2, yq_1; q_1; x \\ yq_1^2 \end{array}\right]}{{}_2\Phi_1\left[\begin{array}{l} q^2, \beta q; q; x \\ \beta q^2 \end{array}\right]} $$

$$ = \frac{\left[xyq_1, \frac{1}{xy}; q_1\right]_\infty [xq, q/x, \beta q^2, q/\beta; q]_\infty [q_1;q_1]^2_\infty (1-q)}{[xq_1, q_1/x, q_1/y, yq_1^2; q_1]_\infty [x\beta q, 1/x\beta; q]_\infty [q;q]^2_\infty (1-q_1)}. \tag{2.1} $$

Again taking $y = \dfrac{1}{q_1}, \beta = -\dfrac{1}{q}$ in (2.1) and making use of definition (1.1), we have

$$ \frac{\sum_{n=0}^{\infty} \dfrac{(1-q_1^{n+1})}{(1+q_1^n)} x^n}{\sum_{n=0}^{\infty} \dfrac{(1-q^{n+1})}{1+q^n} x^n} = \frac{[-x, -q_1/x; q_1]_\infty [xq, q/x; q]_\infty}{[xq_1, q_1/x; q_1]_\infty [-x, -q/x; q]_\infty} $$

$$ \times \frac{\eta^4(z_1)\eta^2(2z)}{\eta^2(2z_1)\eta^4(z)} \tag{2.2} $$

where $q_1 = e^{2\pi i z_1}$ and $q = e^{2\pi i z}$.

Replacing q, q_1 by q^2, q_1^2 respectively in (2.2) and then putting $x = q_1$, we have

$$ \frac{\sum_{n=0}^{\infty} \dfrac{(1-q_1^{2n+2})}{(1+q_1^{2n})} q_1^n}{\sum_{n=0}^{\infty} \dfrac{(1-q^{2n+2})}{1+q^{2n}} q_1^n} = \frac{(1-q_1)\eta^{10}(2z_1)\eta^2(4z)[q_1q^2, q^2/q_1; q^2]_\infty}{\eta^4(z_1)\eta^4(4z_1)\eta^4(2z)[-q_1, -q^2/q_1; q^2]_\infty} \tag{2.3} $$

Taking $y = \dfrac{1}{q_1}$ in (1.1), we get

$$ {}_2\Phi_1 \left[\begin{array}{l} q^2, \beta q; q; x \\ \beta q^2 \end{array} \right] = \frac{[x\beta q, 1/x\beta; q]_\infty [q;q]_\infty^2}{(1-q)[x, q/x, \beta q^2, q/\beta; q]_\infty}. \tag{2.4} $$

Replacing q by q^3 and then taking $\beta = q^{-2}$ and $x = -q^6$ in (2.4), we get

$$ \sum_{n=0}^{\infty} \frac{(1-q^{3n+3})(-)^n q^{6n}}{(1-q^{3n+1})} = \left(\frac{1-q^2}{2q^2} \right) \frac{\eta^6(3z)\eta(2z)}{\eta^2(z)\eta^3(6z)}. \tag{2.5} $$

Putting $\beta = -\dfrac{1}{q}$ in (2.4), we have

$$ \frac{2}{1+q} \sum_{n=0}^{\infty} \left(\frac{1-q^{n+1}}{1+q^n} \right) x^n = \frac{[-x, -q/x; q]_\infty}{[x, q/x; q]_\infty} \frac{\eta^4(z)}{\eta^2(2z)}. \tag{2.6} $$

Again, replacing q by q^2 and then taking $x = q$ in (2.6), we find

$$ \left(\frac{2}{1+q^2} \right) \sum_{n=0}^{\infty} \left(\frac{1-q^{2n+2}}{1+q^n} \right) q^n = \frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)}. \tag{2.7} $$

A number of other interesting results can also be scored.

3. Part II

In the 'Lost Notebook' of Ramanujan [4], he has mentioned following beautiful theorem If $a, b \neq -q^{-n}$, then

$$ \rho(a,b) - \rho(b,a) = \left(\frac{1}{b} - \frac{1}{a} \right) \frac{[q, aq/b, bq/a; q]_\infty}{[-aq, -bq; q]_\infty}, \tag{3.1} $$

where

$$ \rho(a,b) = \left(1 + \frac{1}{b} \right) \sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2} (a/b)^n}{[-aq; q]_n}. \tag{3.2} $$

4. Main Results

(a) In this section we shall establish results involving eta-functions, let,

$$ F(a,b;q) = \rho(a,b) - \rho(b,a) = \left(\frac{1}{b} - \frac{1}{a} \right) \frac{[q, aq/b, bq/a; q]_\infty}{[-aq, -bq; q]_\infty} \tag{4.1} $$

(i) Replacing q by q^2 and then taking $a=-q, b=q$ in (4.1), we get

$$F(-q,q;q^2)=\frac{2(1-q^2)}{q^{4/3}}\frac{\eta^3(4\zeta)}{\eta^2(2\zeta)}. \tag{4.2}$$

(ii) Replacing q by q^2 and then taking $a=q, b=-q$ in (4.1), we get

$$F(q,-q;q^2)=\frac{-2(1-q^2)}{q^{4/3}}\frac{\eta^3(4\zeta)}{\eta^2(2\zeta)}. \tag{4.3}$$

(iii) Comparing (4.2) and (4.3), we get

$$F(q,-q;q^2)=-F(-q,q;q^2). \tag{4.4}$$

(iv) Taking $a=-q$ and $b=q$ in (4.1), we get

$$F(-q,q;q)=\left(\frac{1-q}{2q}\right)\frac{1}{q^{1/24}}\frac{\eta(2\zeta)}{\eta(\zeta)}. \tag{4.5}$$

(v) Replacing q by q^3 and then taking $a=q$ and $b=q^2$ in (4.1), we find

$$F(q,q^2,q^3)=\frac{(1+q)(1+q^2)}{q^{17/8}}\frac{\eta^2(\zeta)\eta(6\zeta)}{\eta(2\zeta)\eta(3\zeta)}. \tag{4.6}$$

(vi) Replacing q by q^3 and then taking $a=-q$ and $b=-q^2$ in (4.1), we find

$$F(-q,-q^2;q^3)=-\frac{(1-q)(1-q^2)}{q^{17/8}}\eta(3\zeta). \tag{4.7}$$

(vii) Replacing q by q^4 and then taking $a=-q^2$ and $b=q^2$ in (4.1), we get

$$\begin{aligned}F(-q^2,q^2;q^4)&=\frac{1}{2q^2}\frac{(q^4;q^4)_\infty(-q^4;q^4)^2_\infty}{(q^6,-q^6;q^4)_\infty}\\&=\frac{(1-q^4)}{2q^2}\frac{(q^8;q^8)^3_\infty}{(q^4;q^4)^2_\infty}\\&=\frac{(1-q^4)}{2q^{8/3}}\frac{\eta^3(8\zeta)}{\eta^2(4\zeta)}.\end{aligned} \tag{4.8}$$

(viii) Replacing q by q^6 and then taking $a=-q^3$ and $b=q^3$ in (4.1), we obtain

$$F(-q^3,q^3;q^6)=\frac{(1-q^6)}{(2q^4)}\frac{\eta^3(12\zeta)}{\eta^2(6\zeta)} \tag{4.9}$$

(ix) Replacing q by q^5 and then taking $a = -q^2$ and $b = q^3$ in (4.1), we get

$$F(-q^2, -q^3; q^4) = -\frac{(1-q^2)(1-q^3)}{(q^3)}(q^5; q^5)_\infty \frac{(q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty} \tag{4.10}$$

Now, using [Andrews and Berndt 1; Corollary (6.2.6) p. 153] in (4.10), we get

$$F(-q^2, q^3; q^5) = -\frac{(1-q^2)(1-q^3)}{q^3}(q^5; q^5)_\infty \left\{ \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\ldots} \right\}. \tag{4.11}$$

5. Another form of the function $\rho(a, b)$

Let us consider the Rogers-Fine identity, viz.

$$\sum_{n=0}^{\infty} \frac{(\alpha; q)_n}{(\beta; q)_n} z^n = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n (\alpha z q/\beta; q)_n \beta^n z^n (1 - \alpha z q^{2n}) q^{n(n-1)}}{(\beta; q)_n (z; q)_{n+1}} \tag{5.1}$$

[Andrews and Berndt 1; (9.11) p. 223]

Putting z/α for z and then taking $\alpha \to \infty$ in (5.1), we find

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n-1)/2} z^n}{(\beta; q)_n} = \sum_{n=0}^{\infty} \frac{(-)^n q^{3n(n+1)/2} (zq/\beta; q)_n \beta^n z^n (1 - zq^{2n})}{(\beta; q)_n} \tag{5.2}$$

Now taking $\beta = -aq$ and $z = aq/b$ in (5.2), we have

$$\rho(a, b) = \sum_{n=0}^{\infty} \frac{(-1/b; q)_{n+1} \left(\frac{a^2}{b}\right)^n q^{n(3n+1)/2} \left(1 - \frac{a}{b} q^{2n+1}\right)}{(-aq; q)_n}$$

$$= \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(-q/b; q)_n \left(\frac{a^2}{b}\right)^n q^{n(3n+1)/2} \left(1 - \frac{a}{b} q^{2n+1}\right)}{(-aq; q)_n} \tag{5.3}$$

which is another form of $\rho(a, b)$.
For $a = b = 1$, (5.3) yields,

$$\rho(1, 1) = 2 \sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}). \tag{5.4}$$

For $a = 1, b = q$, (5.3) yields,

$$\rho(1, q) = 2\left(1 + \frac{1}{q}\right) \sum_{n=1}^{\infty} q^{n(3n-1)/2} (1 - q^n). \tag{5.5}$$

For $a = b = q$, (5.3) yields,

$$\rho(q, q) = 2\frac{(1+q)^2}{q}\sum_{n=0}^{\infty} q^{3n(n+1)/2}\frac{(1-q^{2n+1})}{(1-q^{n+1})}. \tag{5.6}$$

For $a = q, b = 1$, (5.3) yields

$$\rho(q, 1) = 2(1+q)\sum_{n=0}^{\infty} q^{n(3n+5)/2}(1-q^{n+1}). \tag{5.7}$$

From (3.2) and (5.3), we get

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}). \tag{5.8}$$

From (3.2) and (5.5), we get

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n-1)/2}}{(-q;q)_n} = 2\sum_{n=1}^{\infty} q^{n(3n-1)/2}(1-q^{n}). \tag{5.9}$$

From (3.2) and (5.6), we get

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)/2}}{(-q;q)_n} = 2\sum_{n=0}^{\infty} q^{3n(n+1)/2}\frac{(1-q^{2n+1})}{(1-q^{n+1})}. \tag{5.10}$$

From (3.2) and (5.7), we find

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+3)/2}}{(-q;q)_{n+1}} = 2\sum_{n=0}^{\infty} q^{n(3n+5)/2}(1-q^{n+1}). \tag{5.11}$$

6. Part III
Further results involving eta functions

In this section we shall make use of the following results due to Ramanujan, If

$$\rho(a, b, c; q) = \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(c)_n(-)^n q^{n(n+1)/2}\left(\frac{a}{b}\right)^n}{[-aq;q]_n(-c/b;q)_{n+1}}$$

then

$$\rho(a, b, c; q) - \rho(b, a, c; q) = \left(\frac{1}{b}-\frac{1}{a}\right)\frac{[c, aq/b, bq/a, q; q]_\infty}{[-c/a, -c/b, -aq, -bq; q]_\infty} \tag{6.1}$$

Let us suppose

$$F(a,b,c;q) = \frac{\rho(a,b,c;q) - \rho(b,a,c;q)}{\rho(a,b;q) - \rho(b,a;q)} = \frac{[c;q]_\infty}{[-c/a,-c/b;q]_\infty},$$

where

$$\rho(a,b;q) = \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}\left(\frac{a}{b}\right)^n}{[-aq;q]_n} \tag{6.2}$$

(i) Replacing q by q^2 and then taking $c = q^2$ and $a = b = +1$ in (6.2), we get

$$F(+1,+1,q^2;q^2) = \frac{[q^2;q^2]_\infty}{[-q^2;q^2]_\infty^2} = \frac{[q^2;q^2]_\infty^3}{[q^4;q^4]_\infty^2}$$

$$= \frac{q^{1/12}\eta^3(2\tau)}{\eta^2(4\tau)} \tag{6.3}$$

(ii) Replacing q by q^2 and then taking $c = q^2$ and $a = 1, b = -1$ in (6.2), we get

$$F(1,-1,q^2;q^2) = \frac{[q^2;q^2]_\infty}{[-q^2;q^2]_\infty[q^2;q^2]_\infty} = \frac{[q^2;q^2]_\infty}{[q^4;q^4]_\infty}$$

$$= \frac{q^{1/12}\eta(2\tau)}{\eta(4\tau)}. \tag{6.4}$$

(iii) Replacing q by q^2, then taking $c = q^2$, $a = b = -1$ in (6.2), we get

$$F(-1,-1,q^2;q^2) = \frac{[q^2;q^2]_\infty}{[q;q^2]_\infty^2} = \frac{1}{[q^2;q^2]_\infty} = \frac{q^{1/12}}{\eta(2\tau)}. \tag{6.5}$$

(iv) Replacing q by q^2, then taking $c = q^2$, $a = b = -q$ in (6.2), we get

$$F(-q,-q,q^2;q^2) = \frac{[q^2;q^2]_\infty}{[q;q^2]_\infty^2} = \frac{[q^2;q^2]_\infty^3}{[q;q]_\infty^2} = \frac{\eta^3(2\tau)}{q^{1/6}\eta^2(\tau)}. \tag{6.6}$$

(v) Replacing q by q^2, then taking $c = q^2, a = q$ and $b = -q$ in (6.2), we get

$$F(q,-q,q^2;q^2) = \frac{[q^2;q^2]_\infty}{[-q;q^2]_\infty[q;q^2]_\infty} = q^{-1/6}\eta(4\tau). \tag{6.7}$$

(vi) Replacing q by q^2, then taking $c = q^2, a = b = q$ in (6.2), we get

$$F(q,q,q^2;q^2) = \frac{[q^2;q^2]_\infty[-q^2;q^2]_\infty^2}{[-q;q^2]_\infty^2[-q^2;q^2]_\infty^2} = \frac{\eta^2(4\tau)\eta^2(\tau)}{q^{1/6}\eta^3(2\tau)}. \tag{6.8}$$

(vii) From (6.3) and (6.8), we get

$$F(1,1,q^2;q^2)F(q,q,q^2;q^2) = \frac{\eta^2(\tau)}{q^{1/12}}. \tag{6.9}$$

(viii) Taking $c = q, a = b = 1$ in (6.2), we get

$$F(1,1,q;q) = \frac{[q;q]_\infty}{[-q;q]^2_\infty} = q^{1/24}\frac{\eta^3(\tau)}{\eta^2(2\tau)} \tag{6.10}$$

(ix) Taking $c = q, a = 1$ & $b = -1$ in (6.2), we get

$$F(1,-1;q;q) = \frac{[q;q]_\infty}{[q;q]_\infty[-q;q]_\infty} = q^{1/24}\frac{\eta(\tau)}{\eta(2\tau)}. \tag{6.11}$$

(x) Taking $c = q, a = b = -1$ in (6.2), we get

$$F(-1,-1,q;q) = \frac{1}{[q;q]_\infty} = \frac{q^{1/24}}{\eta(\tau)} = \sum_{n=0}^{\infty} p(n)q^n. \tag{6.12}$$

(xi) Taking $c = q, a = b = q$ in (6.2), we get

$$F(q,q,q;q) = \frac{[q;q]^3_\infty}{4[q^2;q^2]^2_\infty} = \frac{q^{1/24}\eta^3(\tau)}{4\eta^2(2\tau)}. \tag{6.13}$$

(xii) Taking $c = q, a = b$ and $b = 1$ in (6.2), we get

$$F(q,1,q;q) = \frac{[q;q]^3_\infty}{2[q^2;q^2]^2_\infty} = \frac{q^{1/24}\eta^3(\tau)}{2\eta^2(2\tau)}. \tag{6.14}$$

(xiii) Taking $c = q, a = q$ and $b = -1$ in (6.2), we get

$$F(q,-1,q;q) = \frac{[q;q]_\infty}{2[q^2;q^2]_\infty} = \frac{q^{1/24}\eta(\tau)}{2\eta(2\tau)}. \tag{6.15}$$

(xiv) Replacing q by q^3 and then taking $c = q^3, a = b = 1$ in (6.2), we get

$$F(1,1,q^3;q^3) = \frac{[q^3;q^3]^3_\infty}{[q^6;q^6]^2_\infty} = q^{1/8}\frac{\eta^3(3\tau)}{\eta^2(6\tau)}. \tag{6.16}$$

(xv) Replacing q by q^3 and then taking $c = q^3, a = 1, b = -1$ in (6.2), we get

$$F(1,-1,q^3;q^3) = \frac{[q^3;q^3]_\infty}{[q^6;q^6]_\infty} = q^{1/8}\frac{\eta(3\tau)}{\eta(6\tau)}. \tag{6.17}$$

(xvi) Replacing q by q^3, then taking $c = q^3, a = b = -1$ in (6.2), we get

$$F(-1, -1, q^3; q^3) = \frac{1}{[q^3; q^3]_\infty} = q^{1/8} \frac{1}{\eta(3\tau)}. \tag{6.18}$$

(xvii) Replacing q by q^3, then taking $c = q^3, a = q$ and $b = q^2$ in (6.2), we get

$$F(q, q^2, q^3; q^3) = \frac{[q^3; q^3]_\infty}{[-q, -q^2; q^3]_\infty} = \frac{\eta(\tau)\eta(6\tau)}{q^{5/24}\eta(6\tau)}. \tag{6.19}$$

(xviii) Replacing q by q^3, then taking $c = q^3, a = -q$ and $b = -q^2$ in (6.2), we get

$$F(-q, -q^2, q^3; q^3) = \frac{[q^3; q^3]_\infty}{[q, q^2; q^3]_\infty} = \frac{\eta^2(3\tau)}{q^{5/24}\eta(\tau)}. \tag{6.20}$$

(xix) Replacing q by q^3, then taking $c = q^3, a = q^3$ and $b = q^3$ in (6.2), we get

$$F(q^3, q^3, q^3; q^3) = \frac{[q^3; q^3]_\infty^3}{4[q^6; q^6]_\infty^2} = q^{1/8} \frac{\eta^3(3\tau)}{4\eta^2(6\tau)}. \tag{6.21}$$

(xx) Replacing q by q^3, then taking $c = q^3, a = q^3$ and $b = 1$ in (6.2), we get

$$F(q^3, 1, q^3; q^3) = \frac{[q^3; q^3]_\infty^3}{2[q^6; q^6]_\infty^2} = q^{1/8} \frac{\eta^3(3\tau)}{2\eta^2(6\tau)}. \tag{6.22}$$

(xxi) Replacing q by q^3, then taking $c = q^3, a = q^3$ and $b = -1$ in (6.2), we get

$$F(q^3, -1, q^3; q^3) = \frac{[q^3; q^3]_\infty}{2[q^6; q^6]_\infty} = q^{1/8} \frac{\eta(3\tau)}{2\eta(6\tau)}. \tag{6.23}$$

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