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# ON CERTAIN INTEGRALS INVOLVING PRODUCT OF HYPERGEOMETRIC FUNCTION AND H-FUNCTION

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**Abstract:** In this paper we have evaluated an integral involving the product of Gaussian's hypergeometric function and H-function of two variables.

**Keywords and Phrases:** H-function of two variables and hypergeometric function.

**2010 Mathematics Subject Classification:** 30C45.

## 1. Introduction

The main aim of this paper is to evaluate an integral involving the product of Gaussian's hypergeometric series and H-function of two variables. Interesting special cases of this integral have also been discussed.

The result established here are of a very general nature and include several known results in the literature.

## 2. Notations and Definitions

The H-function of two variables is defined as [2].

$$H^{n,v_1,v_2,m_1,m_2}_{p,[t,t'],s[q;q']}\left[\begin{array}{c|l} x & \{(\varepsilon_p);(e_p)\} \\ & \{(\gamma_t);c_t\};\{(\gamma'_t);c'_{t'}\} \\ y & \{(\delta_s);d_s\} \\ & \{(\beta_q);b_q\};\{(\beta'_{q'});b'_{q'}\} \end{array}\right]$$

$$= \frac{-1}{4\pi^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\Phi(\xi+\eta)\Psi(\xi,\eta)x^{\xi}.y^{\eta}.d\xi.d\eta. \quad (2.1)$$

where

$$\Phi(\xi+\eta)=\frac{\Gamma[(1-\varepsilon_n)+(e_n)(\xi+\eta)]}{\Gamma[(\varepsilon_{n+1,p})-(e_{n+1,p})(\xi+\eta)]\Gamma[(\delta_s)+(d_s)(\xi+\eta)]} \tag{2.2}$$

$$\Psi(\xi,\eta)=\frac{\Gamma[(\gamma_{v_1})+(c_{v_1}\xi)]\Gamma[(\gamma'_{v_2})+(c'_{v_2}\eta)]\Gamma[(\beta_{m_1})-b_{m_1}\xi]}{\Gamma[(1-\gamma_{v_1+1,t})-(c_{v_1+1,t})\xi]\Gamma[(1-\gamma'_{v_2+1,t'})-(c'_{v_2+1,t})\eta]}$$
$$\times\frac{\Gamma[(\beta'_{m_2})-(b'_{m_2})\eta]}{\Gamma[(1-\beta_{m_1+1,q})+(b_{m_1+1,q})\xi]\Gamma[(1-\beta'_{m_2+1,q'})+(b'_{m_2+1,q})\eta]} \tag{2.3}$$

and $0\le n\le p; 0\le v_1\le t; 0\le v_2\le t'; 0\le m_1\le q; 0\le m_2\le q'$.

The sequence of parameters $\{(\beta_{m_1}),(b_{m_1})\}$, $\{(\beta'_{m_2}),(b'_{m_2})\}$, $\{(\gamma_{v_1}),(c_{v_1})\}$, $\{(\gamma'_{v_2})$ $,(c'_{v_2})\}$ and $\{(\varepsilon_n),(e_n)\}$ are such that none of the poles of integrand coincide. The paths of integration are indented, if necessary, in such a manner that all the poles due to $\Gamma\{(\beta_{m_1})-(b_{m_1})\xi\}$ and $\Gamma\{(\beta'_{m_2})-(b'_{m_2})\eta\}$ lie to the right and those due to $\Gamma\{(\gamma_{v_1})+(c_{v_1})\xi\}$ and $\Gamma\{(\gamma'_{v_2})+(c'_{v_2})\eta\}$ and $\Gamma\{1-(\varepsilon_n)+(e_n)(\xi+\eta)\}$ lie to the left of the imaginary axis.

## 3. Main Results

Here we establish the following

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\delta-\alpha-1}\ {}_2F_1\left[\begin{matrix}\beta,-t;\lambda\\ 1+\alpha+\beta-\delta-t\end{matrix}\right]\times$$

$$\times H^{n_1,m_2,n_2,m_3,n_3}_{p_1,[p_2;p_3],q_1[q_2;q_3]}\left[\begin{matrix}\lambda^l\\ \\ \lambda^m\end{matrix}\left|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}\\ (c_j;\gamma_j)_{1,p_2}:(e_j;E_j)_{1,p_3}\\ (b_j;\beta_j,B_j)_{1,q_1}\\ (d_j;\delta_j)_{1,q_2}:(f_j;F_j)_{1,q_3}\end{matrix}\right.\right]d\lambda$$

$$=\frac{\Gamma(\delta-\alpha+t)}{(\delta-\alpha-\beta)_t}H^{2+n_1,m_2,n_2,m_3,n_3}_{2+p_1,[p_2;p_3],q_1+2,[q_2;q_3]}$$

$$\left[\begin{matrix}x\\ \\ y\end{matrix}\left|\begin{matrix}(1-\delta;l,m),(1-\delta+\beta-t;l,m),(a_j;\alpha_j,A_j)_{1,p_1}\\ (c_j;\gamma_j)_{1,p_2}:(e_j;E_j)_{1,p_3}\\ (b_j;\beta_j,B_j)_{1,q_1}(\delta-\beta;l,m),(\delta+t;l,m)\\ (d_j;\delta_j)_{1,q_2}:(f_j;F_j)_{1,q_3}\end{matrix}\right.\right] \tag{3.1}$$

**Proof.**

Let us consider the integral

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\delta-\alpha-1}\ {}_2F_1\left[\begin{matrix}\alpha',\beta';\lambda x\\ \gamma'\end{matrix}\right]\times$$

$$\times H^{n_1,m_2,n_2,m_3,n_3}_{p_1,[p_2;p_2],q_1[q_2;q_3]}\left[\begin{array}{c|l} \lambda^l x & (a_j;\alpha_j,A_j)_{1,p_1} \\ & (c_j;\gamma_j)_{1,p_2}:(e_j;E_j)_{1,p_3} \\ \lambda^m y & (b_j;\beta_j,B_j)_{1,q_1} \\ & (d_j;\delta_j)_{1,q_2}:(f_j;F_j)_{1,q_3} \end{array}\right] d\lambda$$

We replace the H-function on the left hand side of above equation by its equivalent double Mellin Barne's type contour integral and also the hypergeometric function by it's equivalent series and interchanging the order of integration, we have

$$-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\Phi(\xi+\eta)\Phi_2(\xi)\Phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta\times$$

$$\times\sum_{n=0}^{\infty}\frac{(\alpha')_n(\beta')_n x^n}{(1)_n(\gamma')_n}\int_0^1\lambda^{\alpha+n+l\xi+m\eta-1}(1-\lambda)^{\delta-\alpha-1}d\lambda$$

where $\Phi_1$, $\Phi_2$ and $\Phi_3$ are same as in the definition of H-function and using beta integral we have,

$$-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\Phi(\xi+\eta)\Phi_2(\xi)\Phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta\times$$

$$\times\sum_{n=0}^{\infty}\frac{(\alpha')_n(\beta')_n x^n}{(1)_n(\gamma')_n}\frac{\Gamma(\alpha+n+l\xi+m\eta)\Gamma(\delta-\alpha)}{\Gamma(\delta+l\xi+m\eta+n)},$$

provided $Re(\alpha+l\xi+m\eta)>0$ and $Re(\delta-\alpha)>0$.
Now, using (2.1) here, we get

$$-\Gamma(\delta-\alpha)\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\Phi(\xi+\eta)\Phi_2(\xi)\Phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta\times$$

$$\times\frac{\Gamma(\alpha+l\xi+m\eta)}{\Gamma(\delta+l\xi+m\eta)}\sum_{n=0}^{\infty}\frac{(\alpha')_n(\beta')_n(\alpha+l\xi+m\eta)_n x^n}{(1)_n(\gamma')_n(\delta+l\xi+m\eta)}d\xi d\eta$$

$$=-\Gamma(\delta-\alpha)\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\Phi_1(\xi+\eta)\Phi_2(\xi)\Phi_3(\eta)x^{\xi}y^{\eta}\times$$

$$\times\frac{\Gamma(\alpha+l\xi+m\eta)}{\Gamma(\delta+l\xi+m\eta)}\;{}_3F_2\left[\begin{array}{l}\alpha+l\xi+m\eta,\alpha',\beta';x\\ \delta+l\xi+m\eta,\gamma'\end{array}\right]d\xi d\eta$$

Taking $x=1$ $\alpha'=\beta$, $\beta'=-t$ and $\gamma'=1+\alpha+\beta-\delta-t$ in the above, we get

$$-\Gamma(\delta-\alpha)\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\Phi_1(\xi+\eta)\Phi_2(\xi)\Phi_3(\eta)x^{\xi}y^{\eta}\times$$

$$\times\frac{\Gamma(\alpha+l\xi+m\eta)}{\Gamma(\delta+l\xi+m\eta)}\ {}_3F_2\left[\begin{array}{l}\alpha+l\xi+m\eta,\beta,-t;1\\ \delta+l\xi+m\eta,1+\alpha+\beta-\delta-t\end{array}\right]d\xi d\eta$$

Now, evaluating ${}_3F_2$ with the help of Saalschütz summation formula we get

$$-\frac{\Gamma(\delta-\alpha)(\delta-\alpha)_t}{(\delta-\alpha-\beta)_t}\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\Phi_1(\xi+\eta)\Phi_2(\xi)\Phi_3(\eta)x^{\xi}y^{\eta}\times$$

$$\times\frac{\Gamma(\delta+l\xi+m\eta)\Gamma(\delta-\beta+l\xi+m\eta)}{\Gamma(\delta+t+l\xi+m\eta)\Gamma(\delta-\beta+t+l\xi+m\eta)}d\xi d\eta.$$

Replacing the double Mellin Barne's contour integral in its equivalent H-function

$$=\frac{\Gamma(\delta-\alpha+t)}{(\delta-\alpha-\beta)_t}H^{2+n_1,m_2,n_2,m_3,n_3}_{p_1+2,[p_2;p_3],2+q_1,[q_2;q_3]}$$

$$\left[\begin{array}{c|l} x & (1-\delta;l,m),(1-\delta+\beta-t;l,m),(a_j;\alpha_j,A_j)_{1,p_1}\\ & (c_j;\chi_j)_{1,p_2}:(e_j;E_j)_{1,p_3}\\ y & (b_j;\beta_j,B_j)_{1,q_1}(\delta-\beta;l,m),(\delta+t;l,m)\\ & (d_j;\delta_j)_{1,q_2}:(f_j;F_j)_{1,q_3}\end{array}\right]$$

which is the required result (5.1).

## 4. Particular Cases

**(a)** On the specializing the parameter particular cases (3.1) by taking $a_j=A_j=1$, $b_j=B_j=c_j=E_j=d_j=F_j=1$ and also $l=m=1$; the result of (3.1(a)) reduces to that of Meijer's G-function of two variables, we have

**(i)**

$$\int_0^1\lambda^{\alpha-1}(1-\lambda)^{\delta-\alpha-1}\ {}_2F_1\left[\begin{array}{l}\beta,-t;\lambda\\ 1+\alpha+\beta-\delta-t\end{array}\right]\times$$

$$\times G^{n_1,m_2,n_2,m_3,n_3}_{p_1,[p_2;p_3],q_1[q_2;q_3]}\left[\begin{array}{c|l}\lambda^l x & (a_{p_1})\\ & (c_{p_2});(e_{p_3})\\ \lambda^m y & (b_{p_1})\\ & (d_{p_2});(d_{p_3})\end{array}\right]$$

$$=\frac{\Gamma(\alpha-\beta+t)}{(\delta-\alpha-\beta)_t}G^{2+n_1,m_2,n_2,m_3,n_3}_{2+p_1,[p_2;p_3],q_1+2,[q_2;q_3]}$$

$$\left[\begin{array}{c|l} x & (1-\delta),(1-\delta+\beta-t),(a_{p_1})\\ & (c_{p_2});(e_{p_3})\\ y & (b_{q_1}),(\delta-\beta),(\delta+t)\\ & (d_{q_2});(d_{q_3})\end{array}\right]$$

**(b)** Next, if we put $l = m$ in (3.1), we get, after some simplification,
**(ii)**

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\delta-\alpha-1}\ {}_2F_1\left[\begin{array}{l}\beta, -t; \lambda\\ 1+\alpha+\beta-\delta-t\end{array}\right]\times$$

$$\times H^{n_1,m_2,n_2,m_3,n_3}_{p_1,[p_2;p_2],q_1[q_2;q_3]}\left[\begin{array}{c|l}\lambda^l x & (a_j;\alpha_j,A_j)_{1,p_1}\\ & (c_j;\gamma_j)_{1,p_2} : (e_j;E_j)_{1,p_3}\\ \lambda^m y & (b_j;\beta_j,B_j)_{1,q_1}\\ & (d_j;\delta_j)_{1,q_2} : (f_j;F_j)_{1,q_3}\end{array}\right] d\lambda$$

$$= \frac{\Gamma(\delta-\alpha+t)}{(\delta-\alpha-\beta)_t} H^{2+n_1,m_2,n_2,m_3,n_3}_{2+p_1,[p_2;p_3],q_1,[q_2;q_3]}$$

$$\left[\begin{array}{c|l}x & (1-\delta;l,l),(1-\delta+\beta-t;l,l),(a_j;\alpha_j,A_j)_{1,p_1}\\ & (c_j;\gamma_j)_{1,p_2} : (e_j;E_j)_{1,p_3}\\ y & (\delta-\beta;l,l),(\delta+t;l,l)(b_j;\beta_j,B_j)_{1,q_1}\\ & (d_j;\delta_j)_{1,q_2} : (f_j;F_j)_{1,q_3}\end{array}\right].$$

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