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N-HIGH SUBMODULES AND h-TOPOLOGY

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Abstract: A submodule K of M is N-high if K is maximal with the property $K \cap N = 0$. where N is a submodule of M. In this paper we study N-high submodules in the light of h-topopogy. In h-topology, $H_k(M)$, $k = 1,2,\ldots,\infty$ form a neighbourhood system for zero. We characterize the submodules N of a h-reduced S_2-model M for which all N-high submodules are bounded. We further characterize the submodule N of the same module M for which all h-pure N-high submodules are bounded,

1. Introduction

All the rings considered here are associative, with unity and the modules are torsion, unital, tight S_2-modules. M i.e. they satisfy the following conditions :

(i) Every finitely generated submodule of every homomorphic image of M is the different sum of uniserial modules.

(ii) For any two uniserial modules U and V of homomorphic image of M, for any $W \subseteq U$ any nonzero homomorphism $f : W \to V$ may be extended to a homomorphism $g : U \to V$ provided that the composition length $d(U/W) \leq d(V/f(W))$.

For the basic definitions and results we refer to [2,3,4].

K.Benabdallah and S. Singh [1] proved that any countably generated submodule of a S_2-module M is contained in a contably generated h-pure submodule of M. Since the basic submodule B of M is of the form B= $\oplus_{i=1}^{\infty} B_i$ i.e. B is countably generated. M will have a proper basic submodule if M is unbounded. If M does not have a proper h-pure submodule containing N, then all N-high submodules of M are boubded, provided that M is h-reduced. M should be h-reduced because for an unbounded N-high submodule of M. K contains a proper basic submodule B and M/B = K/B $\oplus$ T/B imply that T is h-pure in M.

These facts motivate the following :

2. Results

Theorem 2.1 : Let N be a submodule of a h-reduced module M then all N-high submodules of M, are bounded iff $(M_k + N)/N$ is finite for some k where $M_k = Soc(H_k(M))$.

Proof : Suppose $(M_k+N)/N$ is finite for some k and T is a N-high submodule of M. Now Soc $(T \cap H_k(M)) = T \cap M_k$ is finite and $(T \cap H_k(M)) / N$ is also finite, otherwise M can not be h-reduced.

Now $(T + H_k(M))/ H_k(M) \cong T/(T \cap H_k(M))$ and $T/(T \cap H_k(M))$ is bounded $\Rightarrow$ T is also bounded.

Conversely suppose $(M_k + N) / N$ is infinite for all k i.e. $\exists$ a sequence of uniform elements $\{x_i + N\}$ of (Soc (M) + N)/N with increasing heights such that the sum $\oplus(x_i + N)R$ is direct. Let $y_i \in M$ such that $d(y_i R / x_i R)=n_i$ where $H(x_i) = n_i \Rightarrow T = \oplus y_i R$ is unbounded N-high submodule of M.

To characterize the submodules of M in the light of h-topology, we prove the following :

Lemma 2.2 : The intersection of the family of N-high submodules of a S_2-module M is zero, whenever N is nontrivial submodule of M.

Proof. Consider the family $\mathcal{T}$ of all N-high submodules of M and $T \in \mathcal{T}$. For $y \in N$, $x \in T$, $x+y \notin T$ and N. Now $(x+y) R \cap N=0$, therefore $(x+y)R$ may be extended to a N-high submodule K of M and $y \notin K$.

Theorem 2.3: Let $N(\neq 0)$ be a submodule of M then all N-high submodules of M are complete with respect to h-topology of M if and only if they are bounded and M is free from the elements of infinite height.

Proof. Suppose all N-high submodules are complete i.e. M^1 is contained in all N-high submodules as it is the completion of {0} and by lemma 1.2 $M^1=\{0\}$.

If there exists a N-high submodule $T \subset M$ which is not bounded i.e. T contains a proper basic submodule B such that T/B is h-divisible. Now $M/B=T/B \oplus L/B$ $(N \subset L)$. As $(M/B)^1 \neq 0$ and $(N \oplus B)/B \cap (T/B)=0$, we get a $(N \oplus B)/B$-high submodule $K/B \subseteq M/B$ which is not complete in M/B. K is N-high in M and K is not complete $\Rightarrow$ T bounded.

Conversely suppose $M^1=0$ and all N-high submodules are bounded i.e. $\exists$ an integer K such that $(\text{Soc } H_k(m)+N)/N$ is finite. This implies that $\bigcap_{k=0}^{\infty} (T + H_k(M)) = T$ or T is complete.

Corollary 2.4. Let N be a submodule of a S_2-module M and K be a submodule of M such that $K\cap N=0$. Then all N-high submodules T of M containing K are complete if and only if $(M/K)^1=0$ and T/K is bounded for every T.

Proof. T is a N-high submodule of M containing K if and only if T/K is a (N+K)/K-high submodule of M/K and T is complete in M if and only if T/K is complete in M/K. Now $(M/K)^1=0$ and T/K is bounded for every T.

Theorem 2.5. Let M be a S_2-module such that M/K is a direct sum of uniserial modules for all N-high submodules K of M, then M is a direct sum of uniserial modules.

Proof. Since M/K is a direct sum of uniserial modules for every N-high submodule K i.e. $(M/K)^1=0$ and K is complete $\Rightarrow M^1=0$ and K is bounded.

Now Soc (M)=Soc(N)$\oplus$S for some S and there exists k such that $S \cap$ Soc $(H_k(M))=0$. Let Soc(K)=S, where K is a N-high submodule of M, then $H_k(M)\cap K=0$. Since $(H_k(M)+K)/K \cong H_k(M)$ and the submodule of a decomposable module is also decomposable, therefore $(H_k(M)+K)/K$ is decomposable and $H_k(M)$ is decomposable, implying M is also decomposable.

Now we are able to state that any S_2-module M, with a non-trivial submodule N such that M/K is bounded for an N-high submodule K of M is bounded.

Socles of modules play a very significant role because a submodule $K \subset M$ is N-hign if and only if K is Soc (N)-high. Now we characterize the subsocles S of a h-reduced S_2-module for which all h-pure. S-high modules are bounded.

Theorem 2.6. Let S be a subsocle of a h-reduced S_2-module M, then all h-pure, S-high submodules of M are bounded if and only if $(M_k+S)/S$ is finite for some finite K where $M_k=Soc(H_k(M))$.

Proof. Suppose $(M_k+S)/S$ is infinite for every k. Then we may get a sequence $\{x_n\}$ of uniform elements of M_k-S with increasing heights. If $n_i=H(x_i)$ then there exists $y_i \in M$ such that $d(y_iR/x_iR)=n_i$. The submodule T generated by these y_i's is unbounded and h-pure such that $S \cap T = 0$. The Proof of the converse is same as in Theorem 1.1.

Now we say that the all h-pure, S-high submodules of a S_2-module M are bounded if and only if $(M_k+S)/S$ is finite for some k and $M^1 \subset Soc(S)$.

Lemma 2.7. The intersection of the family of h-pure S-high submodules of M is Zero, where S is a nontrivial subsocle of M.

Proof. Let K be an h-pure, S-high submodule of M and x be any uniform element of K.

Case (i) Suppose there exist an element $y \in S$ and a +ve integer K such that $x + y \in H_k(M)$ and $x + y \notin H_{k+1}(M)$. The submodule A of $H_k(M)$ generated by the element of height k may be embedded in a summand Q of M [Th. 3.2].

If $Q \cap S = 0$, then Q may be extended to a h-pure, S-high submodule T such that $x + y \in T$ and $x \notin T$. If $Q \cap S \neq 0$ then $Soc(Q) = L \oplus (Q \cap S)$ for some L. Since L is also h-pure containing x + y, it can be extended to a h-pure S-high submodule L' such that $x \notin L'$.

Case (ii) Let $x + y \in M^1$ for every $y \in S$ i.e. $H(x) = H(y) = \infty \ \forall y \in S$ and $S \subseteq M^1$ implying that all S-high submodules of M are h-pure.

Theorem 2.8. Let S be a subsocle of a S_2-module M. Then all h-pure S-high submodules of M are complete with respect to the h-topology of M if and only if they are bounded and $M^1 = 0$.

Proof. Suppose all h-pure, S-high submodules are complete then $M^1 = 0$ as M^1 is contained in every complete submodule of M. If K is a complete, h-pure, unbounded and S-high submodule of M then K contains a basic submodule B such that $(K/B)^1 \neq 0$ and hence $(M/B)^1 \neq 0$. Now there exists a h-pure $(S \oplus B)/B$-high submodule T/B which is not complete and we have $((M/B)/(T/B))^1 \neq 0$ implying that T is not complete in M and the result follows, the converse is trivial.

Corollary 2.9. Let S be a nontrivial subsocle of a S_2-module M and K be a h-pure submodule with $K \cap S = 0$. Then all h-pure S-high submodules T containing K are complete in M if and only if $(M/K)^1 = 0$ and T/K is bounded for every K.

Proof. A submodule $T \supset K$ is h-pure and S-high in M if and only if T/K is h-pure $(S \oplus K)$ H-high in M/K and the result follows.

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