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THE ASKEY-WILSON OPERATOR AND THE $_6\Phi_5$ SUMMATION FORMULA

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ABSTRACT : The summation formula $\sum_{k=0}^{n} \frac{(-n)_k (a)_k}{k!(c)_k} = \frac{(c-a)_n}{(c)_n}$ can be proved by expanding each term in the identity $(1-x)^{-\alpha}(1-x)^{-\beta}=(1-x)^{-\alpha-\beta}$ by the binomial theorem, equating coefficients of x^n on both sides and relabelling parameters. The aim of this paper is to use the Askey-Wilson operator D_q and its index lad $D_q^{\alpha} D_q^{\beta} = D_q^{\alpha+\beta}$ to give a similar proof of the summation formula for a terminating, very well poised $_6\phi_5$ series.

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1. INTRODUCTION

The Wilson and Askey-Wilson divided difference operators arise naturally in the theory of the Wilson and Askey-Wilson polynomials, respectively; see [2, pp. 32-36]. Properties and applications of these operators have since been studied by several authors. Askey [1] used the Askey-Wilson operator to give a simple proof of Rogers' connection coefficient formula for the continuous q-ultraspherical polynomials. The Askey-Wilson operator and another operator were used by Kalnins and Miller [7] to derive the orthogonality of the Askey-Wilson polynomials. Ismail [6] gave new proof of the q-Pfaff Saalschütz summation formula and of Sears transformation formula using the Askey-Wilson operator. Magnus [8] showed that the Askey-Wilson operator is the most general of its type.

Cooper [4] has defined fractional powers of the Wilson and Askey-Wilson operators. In verifying the index law $D_q^{\alpha}D_q^{\beta}=D_q^{\alpha+\beta}$ for the Askey-Wilson operator, the summation formula for a terminating, very well-poised ${}_6\phi_5$ series is used. This procedure can be reversed. The aim of this article is to show how the summation formula for a terminating, very well-poised ${}_6\phi_5$ series can be discovered and proved using the Askey-Wilson operator. We will also show in the q=1 case how the Wilson operator can be used to find and prove the summation formula for a terminating, very well-poised ${}_5F_4$ series.

2. Notation and definitions

1. Throughout this paper, τ is assumed to be any complex number satisfying Im $\tau > 0$. Put $q=\exp(i\pi\tau)$. Then $|q|<1$.

2. Let f be an even function. The Wilson operator D is defined by

$$Df(x)=\frac{f\left(x+\frac{i}{2}\right)-f\left(x-\frac{i}{2}\right)}{\left(x+\frac{i}{2}\right)^2-\left(x-\frac{i}{2}\right)^2}=\frac{f\left(x+\frac{i}{2}\right)-f\left(x-\frac{i}{2}\right)}{2ix}. \tag{2.1}$$

The denominator factor 2ix is present in order to make $D(x^2)=1$.

3. Let $f=f(x)$ be a function of $x=\cos\theta$. Let $\phi(\theta)=f(\cos\theta)$. The Askey-Wilson operator is defined by

$$D_q f(x)=\frac{\phi\left(\theta+\frac{\pi\tau}{2}\right)-\phi\left(\theta-\frac{\pi\tau}{2}\right)}{\cos\left(\theta+\frac{\pi\tau}{2}\right)-\cos\left(\theta-\frac{\pi\tau}{2}\right)}=\frac{2\left(\phi\left(\theta+\frac{\pi\tau}{2}\right)-\phi\left(\theta-\frac{\pi\tau}{2}\right)\right)}{e^{i\theta}(q^{1/2}-q^{-1/2})(1-e^{-2i\theta})}. \tag{2.2}$$

The denominator factor $\cos(\theta+\pi\tau/2) - \cos(\theta-\pi\tau/2)$ is present in order to make $D_q(x)=D_q(\cos\theta)=1$.

Standard notation for hypergeometric and basic hypergeometric series, e.g., see [5], will be used throughout.

3. Powers of the operators

A calculation gives

$$\begin{aligned} D^2 f(x) &= D(Df(x)) \\ &= D\left[\frac{f(x+i/2)-f(x-i/2)}{2ix}\right] \\ &= \frac{1}{2ix}\left[\frac{f(x+i)-f(x)}{2ix-1} - \frac{f(x)-f(x-i)}{2ix+1}\right] \\ &= \frac{(2ix-2)f(x+i)}{(2ix-2)(2ix-1)2ix} - 2\frac{2ixf(x)}{(2ix-1)2ix(2ix+1)} + \frac{(2ix+2)f(x-i)}{2ix(2ix+1)(2ix+2)}. \end{aligned} \tag{3.1}$$

A similar calculations leads to

$$\begin{aligned} D^3 f(x) &= \frac{(2ix-3)}{(2ix-3)_4} f\left(x+\frac{3i}{2}\right) - 3\frac{(2ix-1)}{(2ix-2)_4} f\left(x+\frac{i}{2}\right) \\ &+ 3\frac{(2ix+1)}{(2ix-1)_4} f\left(x-\frac{i}{4}\right) - \frac{(2ix+3)}{(2ix)_4} f\left(x-\frac{3i}{2}\right). \end{aligned} \tag{3.2}$$

These formulas lead us to suspect the following pattern.

Proposition 1 let γ be a non-negative integer. Then

$$D^\gamma f(x) = \sum_{n=0}^{\gamma} \frac{(-\gamma)_n}{n!} \frac{(2ix-\gamma+2n)}{(2ix-\gamma+n)_{\gamma+1}} f\left(x+\frac{\gamma-2n}{2}i\right). \tag{3.3}$$

Calculations like those above can also be carried out for the Askey-Wilson operator D_q. The resulting pattern appears to be as follows.

Proposition 1q Let γ be a non-negative integer. Then

$$D_q^{\gamma} f(x) = q^{-\frac{1}{2}\binom{\gamma}{2}} \left[\frac{2}{e^{i\theta}(q^{1/2} - q^{-1/2})}\right] s$$

$$\sum_{n=0}^{\gamma} \frac{(q^{-\gamma};q)_n}{(q;q)_n} \frac{(1-e^{-2i\theta} q^{2n-\gamma})}{(q^{n-\gamma} e^{-2i\theta};q)_{\gamma+1}} q^{\gamma n} \phi\left(\theta + \frac{\gamma - 2n}{2}\pi\tau\right). \tag{3.4}$$

Proofs of Propositions and 1 and 1q

Both Proposition 1 and 1q can be shown to be true using induction on γ. We shall give the details only for proposition 1q since the details for Proposition 1 are similar.

It is straightforward to check that (3.4) reduces to a triviality when γ=0 and reduces to (2.2) when γ=1. Now suppose (3.4) is true for some positive integral value of γ. Using this as the inductive hypothesis we have

$$D_q^{\gamma+1} f(x) = D_q (D_q^{\gamma} f(x))$$

$$= \frac{2}{e^{i\theta}(q^{1/2} - q^{-1/2})(1-e^{-2i\theta})} \left[D_q^{\gamma}\phi\big|_{\theta\to\theta+\pi\tau/2} - D_q^{\gamma}\phi\big|_{\theta\to\theta-\pi\tau/2}\right]$$

$$= \frac{2^{\gamma+1} q^{-\frac{1}{2}\binom{\gamma}{2}}}{e^{i(\gamma+1)\theta}(q^{1/2} - q^{-1/2})^{\gamma+1}(1-e^{-2i\theta})}$$

$$\times \left[\frac{1}{q^{\gamma/2}} \sum_{n=0}^{\gamma} \frac{(q^{-\gamma};q)_n (1-e^{-2i\theta} q^{2n-\gamma-1})}{(q;q)_n (q^{n-\gamma-1} e^{-2i\theta};q)_{\gamma+1}} q^{\gamma n} \phi\left(\theta + \frac{\gamma+1-2n}{2}\pi\tau\right)\right.$$

$$\left. - q^{\gamma} \sum_{n=0}^{\gamma} \frac{(q^{-\gamma};q)_n (1-e^{-2i\theta} q^{2n-\gamma+1})}{(q;q)_n (q^{n-\gamma+1} e^{-2i\theta};q)_{\gamma+1}} q^{\gamma(n)} \phi\left(\theta + \frac{\gamma-1-2n}{2}\pi\tau\right)\right]. \tag{3.5}$$

In the second sum, change all of the occurrences of n to n-1. Then

$$D_q^{\gamma+1} f(x) = \frac{2^{\gamma+1} q^{-\frac{1}{2}\binom{\gamma}{2}-\gamma/2}}{e^{i(\gamma+1)\theta}(q^{1/2}-q^{-1/2})^{\gamma+1}(1-e^{-2i\theta})}$$

$$\times\left[\sum_{n=0}^{\gamma} \frac{(q^{-\gamma};q)_n(1-e^{-2i\theta}q^{2n-\gamma-1})}{(q;q)_n(q^{n-\gamma-1}e^{-2i\theta};q)_{\gamma+1}} q^{\gamma(n)}\phi\left(\theta+\frac{\gamma+1-2n}{2}\pi\tau\right)\right.$$

$$\left.-q^{\gamma}\sum_{n=1}^{\gamma+1} \frac{(q^{-\gamma};q)_{n-1}(1-e^{-2i\theta}q^{2n-\gamma-1})}{(q;q)_{n-1}(q^{n-\gamma}e^{-2i\theta};q)_{\gamma+1}} q^{\gamma(n-1)}\phi\left(\theta+\frac{\gamma+1-2n}{2}\pi\tau\right)\right]. \tag{3.6}$$

Both sums above can be extended to $n \in [0,\ldots,\gamma+1]$, the extra terms being zero because $(q^{-\gamma};q)_{\gamma+1}=0$ and $1/(q;q)_{-1}=0$. This gives

$$D_q^{\gamma+1} f(x) = \frac{2^{\gamma+1} q^{-\frac{1}{2}\binom{\gamma}{2}}}{e^{i(\gamma+1)\theta}(q^{1/2}-q^{-1/2})^{\gamma+1}(1-e^{-2i\theta})}$$

$$\times\sum_{n=0}^{\gamma+1} \frac{(q^{-\gamma};q)_{n-1}(1-e^{-2i\theta}q^{2n-\gamma-1})}{(q;q)_n(q^{n-\gamma-1}e^{-2i\theta};q)_{\gamma+2}} q^{\gamma n}\phi\left(\theta+\frac{\gamma+1-2n}{2}\pi\tau\right)$$

$$\times\left\{(1-q^{-\gamma+n-1})(1-q^{n}e^{-2i\theta})-(1-q^{n})(1-q^{n-\gamma-1}e^{-2i\theta})\right\}.$$

The quantity in braces simplifies to

$$q^{n}(1-q^{-\gamma-1})(e-q^{-2i\theta})$$

hence

$$D_q^{\gamma+1} f(x) = \frac{2^{\gamma+1} q^{-\frac{1}{2}\binom{\gamma}{2}}}{e^{i(\gamma+1)\theta}(q^{1/2}-q^{-1/2})^{\gamma+1}}$$

$$\times\sum_{n=0}^{\gamma+1} \frac{(q^{-\gamma-1};q)_n(1-e^{-2i\theta}q^{2n-\gamma-1})}{(q;q)_n(q^{n-\gamma-1}e^{-2i\theta};q)_{\gamma+2}} q^{(\gamma+1)n}\phi\left(\theta+\frac{\gamma+1-2n}{2}\pi\tau\right).$$

This completes the induction and proves conjecture 1q.

4. Summation formulas

Let α and β be non-negative integers. The operator $D_q^{\alpha+\beta}$ can now be thought of in two ways : either by replacing γ by $\alpha+\beta$ in Proposition 1q, or as the result of applying D_q^{α} to D_q^{β}. When the two ideas are combined, the result is the ${}_6\phi_5$ summation theorem for a terminating, very-well poised series. If the same idea is applied to the operator D, the result is the summation theorem for a terminating, very-well poised ${}_5\phi_4$ series.

Proposition 2 Let α, β and n be non-negative integers. Then

$$ {}_5\phi_4\left(\begin{matrix} A, \frac{A}{2}+1, A+n-\beta, & -\alpha, & -n \\ \frac{A}{2}, \quad \beta+1-n, & A+\alpha+1, & A+n+1 \end{matrix} ;1\right) = \frac{(\alpha+\beta+1-n)_n (A+1;q)_n}{(q^{\beta+1-n};q)_n (q^{A+\alpha+1};q)_n}. \quad (4.1)$$

Proposition 2q Let α, β and n be non-negative integers. Then

$$ {}_6\phi_5\left(\begin{matrix} A, q\sqrt{A}, -q\sqrt{A}, & q^{n-\beta}A, & q^{-\alpha}, & q^{-n} \\ \sqrt{A}, \ -\sqrt{A}, & q^{\beta+1-n}, & q^{\alpha+1}A, & q^{n+1}A \end{matrix} ; q, q^{\alpha+\beta+1}\right)$$

$$= \frac{(q^{\alpha+\beta+1-n};q)_n (qA;q)_n}{(q^{\beta+1-n};q)_n (q^{\alpha+1}A;q)_n}. \quad (4.2)$$

Proof

We shall only prove Proposition 2q. The details for Proposition 2 are similar. By Proposition 1q applied twice, we have

$$D_q^{\alpha}(D_q^{\beta}f(x))$$

$$=D_q^{\alpha}\left\{q^{-\frac{1}{2}\binom{\beta}{2}}\left[\frac{2}{e^{i\theta}(q^{1/2}-q^{-1/2})}\right]^{\beta}\sum_{j=0}^{\infty}\frac{(q^{-\beta};q)_j(1-e^{-2i\theta}q^{2j-\beta})}{(q;q)_j(q^{j-\beta}e^{-2i\theta};q)_{\beta+1}}q^{\beta j}\phi\left(\theta+\frac{\beta-2j}{2}\pi\tau\right)\right\}$$

$$=q^{-\frac{1}{2}\binom{\alpha}{2}-\frac{1}{2}\binom{\beta}{2}}\left[\frac{2}{e^{i\theta}(q^{1/2}-q^{-1/2})}\right]^{\alpha+\beta}\sum_{k=0}^{\infty}\frac{(q^{-\alpha};q)_k(1-e^{-2i\theta}q^{2k-\alpha})}{(q;q)_j(q^{k-\alpha}e^{-2i\theta};q)_{\alpha+1}}q^{\alpha j+\beta(k-\alpha/2)}$$

$$\times\sum_{j=0}^{\infty}\frac{(q^{-\beta};q)_j(1-e^{-2i\theta}q^{2j-\beta-\alpha+2k})}{(q;q)_j(q^{j-\beta-\alpha+2k}e^{-2i\theta};q)_{\beta+1}}q^{\beta j}\left(\theta+\frac{\alpha+\beta-2j-2k}{2}\pi\tau\right).$$

Now let n=j+k and m=k, and remember that only finitely many terms in the double infinite series above are non zero, thus terms in the series may be rearranged as we please.

$$D_q^{\alpha}(D_q^{\beta}f(x))$$

$$=q^{-\frac{1}{2}\binom{\alpha}{2}-\frac{1}{2}\binom{\beta}{2}-\frac{\alpha\beta}{2}}\left[\frac{2}{e^{i\theta}(q^{1/2}-q^{-1/2})}\right]^{\alpha+\beta}$$

$$\times\sum_{n=0}^{\infty}(1-e^{-2i\theta}q^{2n-\alpha-\beta})q^{n\beta}\phi\left(\theta+\frac{\alpha+\beta-2n}{2}\pi\tau\right)$$

$$\times\sum_{m=0}^{n}\frac{(q^{-\alpha};q)_m(1-e^{-2i\theta}q^{2m-\alpha})}{(q;q)_m(q^{m-\alpha}e^{-2i\theta})_{\alpha+1}}\frac{(q^{-\beta};q)_{n-m}}{(q;q)_{n-m}}\frac{q^{\alpha m}}{(q^{n+m-\alpha-\beta}e^{-2i\theta})_{\beta+1}}$$

$$=q^{-\frac{1}{2}\binom{\alpha+\beta}{2}}\left[\frac{2}{e^{i\theta}(q^{1/2}-q^{-1/2})}\right]^{\alpha+\beta}\frac{1}{(e^{-2i\theta}q^{1-\alpha};q)_{\alpha}}$$

$$\times\sum_{n=0}^{\infty}\frac{(q^{-\beta};q)_n(1-e^{-2i\theta}q^{2n-\alpha-\beta})}{(q;q)_n(q^{n-\alpha-\beta}e^{-2i\theta})_{\beta+1}}q^{n\beta}\,\phi\left(\theta+\frac{\alpha+\beta-2n}{2}\pi\tau\right)$$

$$\times \sum_{m=0}^{\infty} \frac{(e^{-2i\theta} e^{-\alpha}; q)_m (1 - e^{-2i\theta} q^{2m-\alpha})(q^{n-\alpha-\beta} e^{-2i\theta})}{(q;q)_m (1 - e^{-2i\theta} q^{-\alpha}; q)(q^{\beta+1-n}; q)_m (qe^{-2i\theta})_m} \frac{(q^{-\alpha}; q)_m (q^{-n}; q)_m}{(q^{1+n-\alpha} e^{-2i\theta}; q)_m} (q^{1+\alpha+\beta})^m .$$

We have used

$$(Aq^m; q)_{s+1} = \frac{(q^m A; q)_\infty}{(q^{m+s+1} A; q)_\infty} = \frac{(A;q)_n (q^{s+1} A; q)_m}{(A;q)_n (q^{s+1} A; q)_\infty} = \frac{(q^{s+1} A; q)_m}{(A;q)_m} (A;q)_{s+1}$$

and

$$\frac{(A;q)_{n-m}}{(B;q)_{n-m}} = \frac{(A;q)_n}{(B;q)_n} \frac{(q^{1-n}/B;q)_m}{(q^{1-n}/A;q)_m} (B/A)^m \tag{4.3}$$

in the above. Next, using

$$\frac{1 - e^{-2i\theta} q^{2m-\alpha}}{1 - e^{-2i\theta} q^{-\alpha}} = \frac{(1 - e^{-i\theta} q^{m-\alpha/2})(1 + e^{-i\theta} q^{m-\alpha/2})}{(1 - e^{-i\theta} q^{-\alpha/2})(1 + e^{-i\theta} q^{-\alpha/2})}$$

$$= \frac{(e^{-i\theta} q^{1-\alpha/2}; q)_m (-e^{-i\theta} q^{1-\alpha/2}; q)_m}{(e^{-i\theta} q^{-\alpha/2}; q)_m (-e^{-i\theta} q^{-\alpha/2}; q)_m}$$

we obtain

$$D_q^\alpha (D_q^\beta f(x))$$

$$= q^{-\frac{1}{2}\binom{\alpha+\beta}{2}} \left[\frac{2}{e^{i\theta}(q^{1/2} - q^{-1/2})} \right]^{\alpha+\beta} \frac{1}{(e^{-2i\theta} q^{1-\alpha}; q)_\alpha}$$

$$\times \sum_{n=0}^{\infty} \frac{(q^{-\beta}; q)_n (1 - e^{-2i\theta} q^{2n-\alpha-\beta})}{(q;q)_n (q^{n-\alpha-\beta} e^{-2i\theta}; q)_{\beta+1}} q^{n\beta} \phi\left(\theta + \frac{\alpha+\beta-2n}{2} \pi\tau\right)$$

$$\times {}_6\phi_5 \left(\begin{matrix} e^{-2i\theta} q^{-\alpha}, e^{-i\theta} q^{1-\alpha/2}, -e^{-i\theta} q^{1-\alpha/2}, e^{-2i\theta} q^{n-\alpha-\beta}, \ q^{-\alpha}, \ q^{-n} \\ e^{-i\theta} q^{-\alpha/2}, \ -e^{-i\theta} q^{-\alpha/2}, \ q^{\beta+1-n}, \ qe^{-2i\theta}, q^{n+1-\alpha} e^{-2i\theta} \end{matrix} ; q, q^{\alpha+\beta+1} \right) . \tag{4.4}$$

Since $D_q^{\alpha}(D_q^{\beta}f(x)) = D_q^{\alpha+\beta}f$, this together with equation (3.4) with $\gamma=\alpha+\beta$ gives two different expressions for $D_q^{\alpha+\beta}f$. Equating these two expressions and picking out the coefficients of $\phi\left(\theta+\dfrac{\alpha+\beta-2n}{2}\pi\tau\right)$ gives

$$\frac{(q^{-\alpha-\beta};q)_n(1-e^{-2i\theta}q^{2n-\alpha-\beta})}{(q;q)_n(q^{n-\alpha-\beta}e^{-2i\theta};q)_{\alpha+\beta+1}}q^{(\alpha+\beta)n}$$

$$=\frac{(q^{-\beta};q)_n(1-e^{-2i\theta}q^{2n-\alpha-\beta})q^{n\beta}}{(e^{-2i\theta}q^{1-\alpha};q)_n(q;q)_n(q^{n-\alpha-\beta}e^{-2i\theta};q)_{\alpha+\beta+1}}$$

$$\times{}_6\phi_5\left(\begin{matrix}e^{-2i\theta}q^{-\alpha}, e^{-i\theta}q^{1-\alpha/2}, -e^{-i\theta}q^{1-\alpha/2}, e^{-2i\theta}q^{n-\alpha-\beta}, q^{-\alpha}, q^{-n}\\ e^{-i\theta}q^{-\alpha/2}, -e^{-i\theta}q^{-\alpha/2}, q^{\beta+1-n}, qe^{-2i\theta}, q^{n+1-\alpha}e^{-2i\theta}\end{matrix};q,q^{\alpha+\beta+1}\right).$$

This simplifies to

$${}_6\phi_5\left(\begin{matrix}e^{-2i\theta}q^{-\alpha}, e^{-i\theta}q^{1-\alpha/2}, -e^{-i\theta}q^{1-\alpha/2}, e^{-2i\theta}q^{n-\alpha-\beta}, q^{-\alpha}, q^{-n}\\ e^{-i\theta}q^{-\alpha/2}, -e^{-i\theta}q^{-\alpha/2}, q^{\beta+1-n}, qe^{-2i\theta}, q^{n+1-\alpha}e^{-2i\theta}\end{matrix};q,q^{\alpha+\beta+1}\right)$$

$$=\frac{(q^{-\alpha-\beta};q)_n(e^{-2i\theta}q^{1-\alpha})_{\alpha}(q^{n-\alpha-\beta}e^{-2i\theta};q)_{\beta+1}q^{n\alpha}}{(q^{-\beta};q)_n(q^{n-\alpha-\beta}e^{-2i\theta};q)_{\alpha+\beta+1}}$$

$$=\frac{(q^{-\alpha-\beta};q)_n(q^{1-\alpha}e^{-2i\theta};q)_{\alpha}q^{n\alpha}}{(q^{-\beta};q)_n(q^{n+1-\alpha}e^{-2i\theta};q)_{\alpha}}$$

$$=\frac{(q^{\alpha+\beta+1-n};q)_n(q^{1-\alpha}e^{-2i\theta};q)_n}{(q^{\beta+1-n};q)_n(qe^{-2i\theta};q)_n}.$$

We have used the properties

$$\frac{(X;q)_n}{(Y;q)_n}=\frac{X^n}{Y^n}\frac{(q^{1-n}/X;q)_n}{(q^{1-n}/Y;q)_n}$$

and

$$\frac{(X;q)_s}{(Xq^t;q)_s} = \frac{(X;q)_t}{(Xq^s;q)_t}$$

to obtain the last line. Since θ is arbitrary, we may replace $e^{-2i\theta}e^{-\alpha}$ with a new parameter A. This completes the proof of Proposition 2q.

As functions of α, both sides of equation (4.1) are rational functions. That is, equation (4.1) is an identity of the form

$$\frac{\text{polynomial in } \alpha \text{ of degree} \le n}{\text{polynomial in } \alpha \text{ of degree} \le n} = \frac{\text{polynomial in } \alpha \text{ of degree} \le n}{\text{polynomial in } \alpha \text{ of degree} \le n}.$$

Furthermore, by Proposition 2 we know that (4.1) is true for infinitely many values of α, namely α=1,2,3,… . Therefore (4.1) remains true for any complex value of α, and so the restriction that α be an integer in Proposition 2 can be dropped. By identical reasoning, the condition that β be an integer can also be dropped. Now let B = A + n – β and C = –α in Proposition 2. Then we have proved the following.

Theorem 3 Let n be a non-negative integer. Then

$$ {}_5F_4\left(\begin{matrix} A, A/2+1, & B, & C, & -n \\ A/2, & A+1-B, & A+1-C, & A+n+1 \end{matrix} ;1\right) = \frac{(A+1-B-C)_n (A+1)_n}{(A+1-B)_n (A+1-C)_n}.$$

Similarly, both sides of (4.2) are rational functions of q^{α} that agree for infinitely many values of α. Hence (4.2) is also true for arbitrary values of α, and by the same reasoning, (4.2) is also true for arbitrary values of β. Let B = $q^{n-\beta}A$ and C = $q^{-\alpha}$. Then we have proved the following.

Theorem 3q

$$ {}_6F_5\left(\begin{matrix} A, q\sqrt{A}, -q\sqrt{A}, & B, & C, & q^{-n} \\ \sqrt{A}, -\sqrt{A}, & qA/B, & qA/C, & q^{n+1}A \end{matrix} ;\frac{q^{n+1}A}{BC}\right)$$

$$= \frac{(qA/BC;q)_n (qA;q)_n}{(qA/B;q)_n (qA/C;q)_n}.$$

5. Remarks

Theorem 3 and 3q can also be extended to the case in which n is not an integer. See [3, p.27] and [5, p.36], respectively.

The main purpose of this paper has been to use powers of the operators D and D_q to obtain summation formulas for terminating, very-well poised ${}_5F_4$ and ${}_6\phi_5$ series. We conclude by mentioning one other application of the operators D^n and D_q^n.

If D_q^n is applied to the weight function for the Askey-Wilson polynomials, the result is a terminating, very-well poised ${}_8\phi_7$ series. Apply Watson's q-analogye of Whipple's transformation [5, p.35] to convert the ${}_8\phi_7$ into a ${}_4\phi_3$. Then apply Sears' transformation [5, p.41] to this to obtain the usual basic hypergeometric form of the Askey-Wilson polynomials. The result (after replacing a, b, c, d with $aq^{n/2},\ldots,dq^{n/2}$) is the Rodrigues formula for Askey-Wilson polynomials, as given in [2, equation (5.15)]. The details will appear in [4].

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References

1 R. Askey, Divided difference operators and classical orthogonal polynomials, Rocky Mountain J, Math. (1989), vol. 19, 33-37.

2 R. Askey and J. Wilson, “Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials”, Memoirs of the Amer. Math. Soc. (1985), Number 319.

3 W. Bailey, “Generalized hypergeometric Series”, Cambridge University Press, 1935.

4 S. Cooper, Fractional powers of the Wilson and Askey-Wilson operators, Work in progress.

5 G. Gasper and M. Rahman, “Basic hypergeometric Series”, Cambridge University Press, 1990.

6 M. Ismail, The Askey-Wilson operator and summation theorems, Mathematical analysis, wavelets, and single processing (Cairo, 1994), 171-178, Contemp. Math., 190, Amer. Math. Soc., Providence, RI, 1995.

7 E. Kalnins and W. Miller, Jr., Symmetry techniques for q-series : Askey-Wilson polynomials, Rocky Mountain J. Math. (1989), vol. 19, 223-230.

8 A. Magnus, Associated Askey-Wilson polynomials as Laguerre-Hahn orthogonal polynomials, in “Orthogonal Polynomials and Their Applications”, eds. M. Alfaro et.al., Lecture Notes in Mathematics, Vol. 1329, Springer verlag, Berlin, 1988, pp.261-278.