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On Certain new identities associated with Ramanujan's modular equations

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Abstract: In this paper, making use of certain modular equations due to Ramanujan, an attempt has been made to establish some new P-Q identities.
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1. Introduction, Notations and Definitions:

Ramanujan has recorded several P-Q eta function identities in chapter 25 of his second notebook. Berndt [4] and Berndt and Zhang [6] proved these P-Q identities by employing various modular equations belonging to the classical theory. In this paper, an attempt has been made to establish certain P-Q identities by making use of modular equations due to Ramanujan. Many other mathematicians have used P-Q eta function identities to evaluate Ramanujan's theta functions, Ramanujan's class invariants, Rogers-Ramanujan continued fraction. In this regard the works of Berndt and Chan [7], Berndt, Chan and Zhang [9], Bhargava and Adiga [10] are note worthy.

Let $Z_r = Z(r,x) = {}_2F_1[1/r,(r-1)/r;1;x]$ and

$$q_r = q_r(x) = exp\left(-\pi cosec(\pi/r)\frac{{}_2F_1[1/r,(r-1)/r;1;1-x]}{{}_2F_1[1/r,(r-1)/r;1;x]}\right), \tag{1.1}$$

where $r = 2,3,4$ and 6 and ${}_2F_1[1/r,(r-1)/r;1;x]$ denotes the ordinary hypergeometric functions defined as;

$${}_2F_1[a,b;c;z] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n n!},$$

where $|z| < 1$
Such that

$$(a)_n = a(a+1)(a+2)...(a+n-1), \qquad n \geq 1$$

and $(a)_0 = 1$.
Also, let n be a positive integer and

$$n\frac{{}_2F_1[1/r,(r-1)/r;1;1-\alpha]}{{}_2F_1[1/r,(r-1)/r;1;\alpha]} = \frac{{}_2F_1[1/r,(r-1)/r;1;1-\beta]}{{}_2F_1[1/r,(r-1)/r;1;\beta]}. \tag{1.2}$$

Then a modular equation of degree n in the theory of elliptic functions of signature r is a relation between α and β induced by (1.2). We often say that β is of degree n over α and $m(r) = Z(r,\alpha)/Z(r,\beta)$ is called the multiplier. We also use the notations $Z_1 = Z(r,\alpha)$ and $Z_n = Z_n(r,\beta)$ to indicate that β has degree n over α. Thus the modular equation of signature 2 is the relation between α and β induced by the equations

$$n\frac{{}_2F_1[1/2,1/2;1;1-\alpha]}{{}_2F_1[1/2,1/2;1;\alpha]} = \frac{{}_2F_1[1/2,1/2;1;1-\beta]}{{}_2F_1[1/2,1/2;1;\beta]},$$

where n is a positive integer, called as the degree of the modular equation. The multiplier m is now defined as;

$$\frac{{}_2F_1[1/2,1/2;1;\alpha]}{{}_2F_1[1/2,1/2;1;\beta]}.$$

In this chapter, an attempt has been made to establish certain new identities by making use of certain modular equations due to Ramanujan. We shall also use the following relations which hold in the theory of signature 2.

$$f(q) = \sqrt{Z}2^{-1/6}\left\{\frac{x(1-x)}{q}\right\}^{1/24} \tag{1.3}$$

$$f(-q^2) = \sqrt{Z}2^{-1/3}\left\{\frac{x(1-x)}{q}\right\}^{1/12}, \tag{1.4}$$

[Ramanujan [11]; chapter 17, entry 12 (i) and (iii)] where $f(-q) = [q;q]_\infty = \prod_{r=1}^{\infty}(1-q^r)$.
Following modular equations are needed in our analysis.

(a) If β, γ and δ are of third , eleventh and thirty-third degrees over α respectively, then

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}+\left\{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right\}^{1/8}-\left\{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/8}$$
$$-2\left\{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/12}=\sqrt{mm'} \quad (1.5)$$

and

$$\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8}+\left\{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right\}^{1/8}-\left\{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right\}^{1/8}$$
$$-2\left\{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right\}^{1/12}=3/\sqrt{mm'} \quad (1.6)$$

[Ramanujan [11]; chapter 20, entry 14 (i) and (ii)]

where m and m' are multipliers associated with α, β and γ, δ respectively.

(b) If β, γ are of third and ninth degrees over α respectively, then

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4}+\left\{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right\}^{1/4}-\left\{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/4}=-3m/m' \quad (1.7)$$

and

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4}+\left\{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right\}^{1/4}-\left\{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right\}^{1/4}=m'/m \quad (1.8)$$

[Ramanujan [11]; chapter 20, entry 3 (xii) and (xiii)]

where m and m' are multipliers associated with the pairs α, β and γ, δ respectively.

(c) If β, γ and δ are third and fifteenth degrees over α respectively, then

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}+\left\{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right\}^{1/8}-\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/8}=\sqrt{m'/m} \quad (1.9)$$

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}+\left\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right\}^{1/8}-\left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/8}=-\sqrt{m/m'} \quad (1.10)$$

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}+\left\{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right\}^{1/4}-\left\{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/4}$$
$$-4\left\{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/6}=mm', \quad (1.11)$$

and

$$\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4}+\left\{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right\}^{1/4}-\left\{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right\}^{1/4}$$
$$-4\left\{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right\}^{1/6}=9/mm', \tag{1.12}$$

[Ramanujan [11]; chapter 20, entry 11 (viii) (ix) (x) and (xi)]

where m and m' are multipliers associated with the pairs α, β and γ, δ respectively.

(d) If β, γ and δ are of third, seventh and twenty first degrees over α respectively, then

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4}+\left\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right\}^{1/4}-\left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/4}$$
$$+4\left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/6}=m/m', \tag{1.13}$$

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/4}+\left\{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right\}^{1/4}-\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/4}$$
$$+4\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/6}=m'/m, \tag{1.14}$$

$$\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/8}+\left\{\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right\}^{1/8}-\left\{\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right\}^{1/8}$$
$$-2\left\{\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right\}^{1/12}=\sqrt{z_1z_3/z_7z_{22}}, \tag{1.15}$$

$$\left(\frac{\alpha\beta}{\gamma\delta}\right)^{1/8}+\left\{\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right\}^{1/8}-\left\{\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right\}^{1/8}$$
$$-2\left\{\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right\}^{1/12}=7\sqrt{z_7z_{21}/z_1z_3}, \tag{1.16}$$

[Ramanujan [11]; chapter 20, entry 13 (i) (ii) (iii) and (iv)]

where m and m' are multipliers associated with the pairs α, β and γ, δ respectively.

(e) If β, γ and δ are of fifth, seventh and thirty-fifth degrees over α respectively, then

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}+\left\{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right\}^{1/8}-\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/8}$$

$$+2\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/12} = \sqrt{m'/m}, \tag{1.17}$$

and

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right\}^{1/8} - \left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/8}$$

$$+2\left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/12} = -\sqrt{m/m'}, \tag{1.18}$$

[Ramanujan [11]; chapter 20, entry 18 (vi) and (vii)]
where m and m' are multipliers associated with the pairs α, β and γ, δ respectively.
(f) If β, γ are of third, thirteenth and thirty-ninth degrees over α respectively, then

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left\{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right\}^{1/8} - \left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/8}$$

$$+2\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/12} = \sqrt{m'/m}, \tag{1.19}$$

and

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right\}^{1/8} - \left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/8}$$

$$+2\left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/12} = \sqrt{m/m'}, \tag{1.20}$$

[Ramanujan [11]; chapter 20, entry 19 (iv)]
where m and m' are multipliers associated with the pairs α, β and γ, δ respectively.

2. Main Results

(a) If

$$P = \frac{f(q)f(-q^2)f(q^{11})f(-q^{22})}{q^3 f(q^3)f(-q^6)f(q^{33})f(-q^{66})}$$

$$Q = \frac{f(q)f(q^{11})}{qf(q^3)f(q^{33})}$$

and

$$R = \frac{f(-q^2)f(-q^{22})}{q^2 f(-q^6)f(-q^{66})}$$

then from the modular equations (1.5) and (1.6) we have

$$P^3 - Q^6 + 2P^2Q^2 - 2PQ^4 + 3PQ^3 - P^2Q^3 = 0. \tag{2.1}$$

and

$$P^3 - R^6 + 2P^2R^2 - 2PR^4 + P^2R^3 - 3PR^3 = 0. \tag{2.2}$$

Now, from (2.1) and (2.2) we have another identity

$$(R - Q)(R^2 + Q^2 + 3P) + 3P - P^2 = 0. \tag{2.3}$$

(b) If

$$P = \frac{f^2(q^3)f^2(-q^6)}{q^{1/2}f(q)f(-q^2)f(q^9)f(-q^{18})}$$

$$Q = \frac{f^2(q^3)}{q^{1/6}f(q)f(q^9)}$$

and

$$R = \frac{f^2(-q^6)}{q^{1/3}f(-q^2)f(-q^{18})}$$

then from (1.7) and (1.9) we have

$$P^6 - Q^{12} - 3P^2Q^6 - P^4Q^6 = 0. \tag{2.4}$$

and

$$P^6 - R^{12} + P^4R^6 + 3P^2R^6 = 0. \tag{2.5}$$

From (2.4) and (2.5) we get another identity

$$R^6 - Q^6 - 3P^2 - P^4 = 0. \tag{2.6}$$

(c) (i) If

$$P = \frac{qf(q)f(-q^2)f(q^{15})f(-q^{30})}{f(q^3)f(-q^6)f(q^5)f(-q^{10})}$$

$$Q = \frac{q^{1/3}f(q)f(q^{15})}{f(q^3)f(q^5)}$$

and

$$R = \frac{q^{2/3}f(-q^2)f(-q^{30})}{f(-q^6)f(-q^{10})}$$

then from (1.9) and (1.10) we have

$$P^3 - Q^6 + PQ^3 + P^2Q^3 = 0. \tag{2.7}$$

and

$$P^3 - R^6 - PR^3 - P^2R^3 = 0. \tag{2.8}$$

Now, from (2.7) and (2.8) we have another identity

$$R^3 - Q^3 + P + P^2 = 0. \tag{2.9}$$

(ii) Again, taking

$$P = \frac{f(q)f(-q^2)f(q^5)f(-q^{10})}{q^{3/2}f(q^3)f(-q^6)f(q^{15})f(-q^{30})}$$

$$Q = \frac{f(q)f(q^5)}{q^{1/2}f(q^3)f(q^{15})}$$

and

$$R = \frac{f(-q^2)f(-q^{10})}{qf(-q^6)f(-q^{30})}$$

and using (1.11) and (1.12) we get thee identities,

$$P^6 - Q^{12} + 4P^4Q^4 - 4P^2Q^8 + 9P^2Q^6 - P^4Q^6 = 0. \tag{2.10}$$

and

$$P^6 - R^{12} + 4P^4R^4 - 4P^2R^8 - 9P^2R^6 + P^4R^6 = 0. \tag{2.11}$$

Also, from (2.10) and (2.11) we get,

$$(R^2 - Q^2)(R^4 + Q^4 + 5P^2) + 9P^2 - P^4 = 0. \tag{2.12}$$

(d) (i) If

$$P = \frac{q^{3/2}f(q)f(-q^2)f(q^{21})f(-q^{42})}{f(q^3)f(q^7)f(-q^6)f(-q^{14})}$$

$$Q = \frac{q^{1/2}f(q)f(q^{21})}{f(q^3)f(q^7)}$$

and

$$R = \frac{qf(-q^2)f(-q^{42})}{f(-q^6)f(-q^{14})}$$

then from (1.13) and (1.14) we get

$$P^6 - Q^{12} - 4P^4Q^4 + 4P^2Q^8 + P^2Q^6 - P^4Q^6 = 0. \tag{2.13}$$

and

$$P^6 - R^{12} - 4P^4R^4 + 4P^2R^8 + P^4R^6 - P^2R^6 = 0. \tag{2.14}$$

From (2.13) and (2.14) we get another identity as,

$$R^6 - Q^6 - 4P^2(R^2 - Q^2) + P^2 - P^4 = 0. \tag{2.15}$$

(ii) If

$$P = \frac{q^3 f(q^7) f(q^{21}) f(-q^{14}) f(-q^{42})}{f(q) f(q^3) f(-q^2) f(-q^6)}$$

$$Q = \frac{q f(q^7) f(q^{21})}{f(q) f(q^3)}$$

and

$$R = \frac{q^2 f(-q^{14}) f(-q^{42})}{f(-q^2) f(-q^6)}$$

then from (1.15) and (1.16) we get

$$P^3 - Q^6 + 2P^2Q^2 - 2PQ^4 + PQ^3 - 7P^2Q^3 = 0. \tag{2.16}$$

and

$$P^3 - R^6 + 2P^2R^2 - 2PR^4 - PR^3 + 7P^2R^3 = 0. \tag{2.17}$$

From (2.16) and (2.17) we get another identity,

$$R^3 - Q^3 + 2P(R - Q) + P - 7P^2 = 0. \tag{2.18}$$

(e) If

$$P = \frac{q^3 f(q) f(q^{35}) f(-q^2) f(-q^{70})}{f(q^5) f(q^7) f(-q^{10}) f(-q^{14})}$$

$$Q = \frac{q f(q) f(q^{35})}{f(q^5) f(q^7)}$$

and

$$R = \frac{q^2 f(-q^2) f(-q^{70})}{f(-q^{10}) f(-q^{14})}$$

then using (1.17) and (1.18) we have

$$P^3 - Q^6 - 2P^2Q^2 + 2PQ^4 + PQ^3 + P^2Q^3 = 0. \tag{2.19}$$

and

$$P^3 - R^6 - 2P^2R^2 + 2PR^4 - PR^3 - P^2R^3 = 0. \tag{2.20}$$

From (2.19) and (2.20) we get another modular identity,

$$R^3 - Q^3 + P^2 - P = 0. \tag{2.21}$$

(f) If

$$P = \frac{q^3 f(q) f(q^{39}) f(-q^2) f(-q^{78})}{f(q^3) f(q^{13}) f(-q^6) f(-q^{26})}$$

$$Q = \frac{q f(q) f(q^{39})}{f(q^3) f(q^{13})}$$

and

$$R = \frac{q^2 f(-q^2) f(-q^{78})}{f(-q^6) f(-q^{26})}$$

then from (1.19) and (1.20) we find

$$P^3 - Q^6 - 2P^2Q^2 + 2PQ^4 + PQ^3 - P^2Q^3 = 0. \tag{2.22}$$

and

$$P^3 - R^6 - 2P^2R^2 + 2PR^4 + P^2R^3 - PR^3 = 0. \tag{2.23}$$

From (2.22) and (2.23) we get another identity,

$$(R - Q)(R^2 + Q^2 - P) + P - P^2 = 0. \tag{2.24}$$

3. Proof

As an illustration we give the proof of (2.1), (2.2) and (2.3). From (1.3), (1.4) and the identities (1.5) and (1.6) we have

$$P = \frac{f(q) f(-q^2) f(q^{11}) f(-q^{22})}{q^3 f(q^3) f(-q^6) f(q^{33}) f(-q^{66})} = mm' \left\{ \frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)} \right\}^{1/8},$$

$$Q = \frac{f(q) f(q^{11})}{q f(q^3) f(q^{33})} = \sqrt{mm'} \left\{ \frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)} \right\}^{1/24}$$

and

$$R = \frac{f(-q^2) f(-q^{22})}{q^2 f(-q^6) f(-q^{66})} = \sqrt{mm'} \left\{ \frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)} \right\}^{1/12}.$$

Thus, we have

$$\frac{P}{Q^2} = \left\{ \frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)} \right\}^{1/24}, \qquad \frac{Q^3}{P} = \sqrt{mm'} \tag{3.1}$$

Also,

$$\frac{R^2}{P} = \left\{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right\}^{1/24}, \qquad \frac{P^2}{R^3} = \sqrt{mm'} \tag{3.2}$$

Now from (1.5) and (3.1)

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} + \left\{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right\}^{1/8} - \frac{Q^6}{P^3} - 2\frac{Q^4}{P^2} = \frac{Q^3}{P} \tag{3.3}$$

and from (1.6) and (3.1)

$$\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8} + \left\{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right\}^{1/8} - \frac{P^3}{Q^6} - 2\frac{P^2}{Q^4} = \frac{3P}{Q^3} \tag{3.4}$$

Let $\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} = u$ and $\left\{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right\}^{1/8} = v$ then (3.3) and (3.4) reduces to

$$u + v = \frac{Q^3}{P} + \frac{Q^6}{P^3} + \frac{2Q^4}{P^2} \tag{3.5}$$

and

$$\frac{1}{u} + \frac{1}{v} = \frac{3P}{Q^3} + \frac{P^3}{Q^6} + \frac{2P^2}{Q^4} \tag{3.6}$$

Eliminating u and v from (3.5) and (3.6)

Keeping in mind that $uv\left\{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/8} = \frac{Q^6}{P^3}$ we get

$$\frac{\frac{Q^3}{P} + \frac{Q^6}{P^3} + \frac{2Q^4}{P^2}}{\frac{Q^6}{P^3}} = \frac{3P}{Q^3} + \frac{P^3}{Q^6} + \frac{2P^2}{Q^4}$$

or

$$P^3\left(\frac{Q^3}{P} + \frac{Q^6}{P^3} + \frac{2Q^4}{P^2}\right) = Q^6\left(\frac{3P}{Q^3} + \frac{P^3}{Q^6} + \frac{2P^2}{Q^4}\right)$$

which on simplification gives

$$P^3 - Q^6 + 2P^2Q^2 - 2PQ^4 + 3PQ^3 - P^2Q^3 = 0.$$

which is (2.1)

Again Using (3.2) eliminate α, β, γ and δ from (1.5) and (1.6) we get (2.2). From (2.1) and (2.2) we can easily obtain (2.3) keeping in mind P=QR.

Proofs of other set of identities are similar

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