sSEA. Jour. Math. & Math.Sc.
Vol.1 No.1 (2002), pp.29-38

A CERTAIN CLASS OF MULTIPLE GENERATING FUNCTIONS INVOLVING MITTAG-LEFFLER'S FUNCTIONS

M. Kamarujjama* and M. Khursheed Alam
Department of Applied Mathematics, Z.H. College of Engineering & Technology
Aligarh Muslim University, Aligarh-202002, INDIA
*e-mail : mdkamarujjama@bharatmail.com

(Received : October 18, 2002)

Abstract: A set of certain class of multiple generating function involving Mittag-Leffler's functions E_α, $E_{\alpha,\beta}$ and related function $\phi(\alpha,\beta;z)$ of wright is given. Some interesting (known and new) multiple generating functions are also obtained as special cases.

Key Words : Mittag-Leffler's functions and related functions and related function, Bessel's function and Hyper-Bessel function, Hypergeometric function.

2000 AMS Subject Classification : Primary 33E12 Secondary 33C99.

1. Introduction and Definition

In the usual notation, let ${}_pF_q$ denote a generalized hypergeometric function of one variable with p and q parameters (positive integer or zero), defined by [6; p.42(1)].

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, ..., a_p; \\ b_1, b_2, ..., b_q \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ...(a_p)_n z^n}{(b_1)_n (b_2)_n ...(b_q)_n n!} $$

$$ = {}_pF_q [a_1, ..., a_p; b_1, ..., b_q; z] \tag{1.1} $$

(by $\neq 0,-1,-2,\ldots$, j=1,2,…,q)

where $(a)_n$ is a Pochhammer symbol, defined by

$$(a)_n = \begin{cases} 1 & , \text{if } n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & \text{if } n = 1,2,\ldots \end{cases} . \tag{1.2}$$

The function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad (\alpha>0) \tag{1.3}$$

was introduced by Mittag-Leffler [5] and was investigated systematically by several other authors [3, chapter XVIII]. $E_\alpha(z)$ for $\alpha>0$, furnishes important example of entire functions of finite order $1/\alpha$.

The function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha,\beta>0) \tag{1.4}$$

has property very similar to those of Mittag-Leffler's function (see Wiman [7], Agarwal [1]).

We have

$$E_{\alpha,1}(z) = E_\alpha(z),\ E_1(z) = e^z,\ E_2(z) = \cos hz$$

and

$$E_{1/2}(\sqrt{z}) = \frac{2e^{-z}}{\sqrt{\pi}} Erfc(-z^{1/2}). \tag{1.5}$$

A function intimately connected with $E_{\alpha,\beta}$ is the entire function

$$\phi(\alpha,\beta;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\alpha k + \beta)}, \qquad (\alpha,\beta>0) \tag{1.6}$$

which was used by Wright [8] in the asymptotic theory of partition.

We can verify easily

$$J_v(z) = \left(\frac{z}{2}\right)^v \phi\left(1, v+1; -\frac{z^2}{4}\right),\ J_v(iz) = \left(\frac{iz}{2}\right)^v \phi\left(1, v+1; -\frac{z^2}{4}\right), \tag{1.7}$$

It shows that Wright's function may be regarded as a kind of generalized Bessel function $J_v(z)$, defined by

$$J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{v+2k}}{\Gamma(v+k+1)k!}, |z| < \infty, \tag{1.8}$$

which has the particular attention in the diverse field of physics and engineering.

An interesting generating function, due to Humbert [4], is recalled here in the following form :

$$\exp\left[\frac{z}{3}\left(x+y \pm \frac{1}{xy}\right)\right] = \sum_{m,n=-\infty}^{\infty} x^m y^n \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\, {}_0F_1\left[-; m+1; n+1; \pm\left(\frac{z}{3}\right)^3\right], \tag{1.9}$$

where the Hyper-Bessel function $J_{m,n}(z)$ and modified Hyper-Bessel function $I_{m,n}(z)$ of order 2 are defined by

$$J_{m,n}(z) = \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\, {}_0F_2\left[-; m+1, n+1; -\left(\frac{z}{3}\right)^3\right] \tag{1.10}$$

and $$I_{m,n}(z) = \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\, {}_0F_2\left[-; m+1, n+1; -\left(\frac{z}{3}\right)^3\right], \text{ respectively.} \tag{1.11}$$

2. Generating function involving Mittag-Leffler's functions

Result-I

If $E_{\alpha,\beta}(z)$ is defined by equation (1.4) then

$$E_{\alpha_1,\beta_1}\left(\frac{zx}{3}\right)E_{\alpha_2,\beta_2}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}\left(\frac{-z/3}{xy}\right)=\sum_{m,n=-\infty}^{\infty}x^m y^n \;{}^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3}J_{m,n}(z), \tag{2.1}$$

where ${}^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3}J_{m,n}(z)$

$$=\left(\frac{z}{3}\right)^{m+n}\sum_{k=0}^{\infty}\frac{(-1)^k (z/3)^{3k}}{\Gamma(\alpha_1(m+k)+\beta_1)\Gamma(\alpha_2(n+k)+\beta_2)\Gamma(\alpha_3 k+\beta_3)}, \tag{2.2}$$

provided that both sides of equation (2.1) exist.

Proof of Result (2.1) :

If the function

$$V(x,y,z)=E_{\alpha_1,\beta_1}\left(\frac{zx}{3}\right)E_{\alpha_2,\beta_2}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}\left(\frac{-z/3}{xy}\right)$$

is expanded by the definition (1.4), we have

$$V=\sum_{k=0}^{\infty}\frac{(-1)^k}{(\alpha_3 k+\beta_3)}\sum_{i=0}^{\infty}\frac{(-1)^{i-k}}{(\alpha_1 i+\beta_1)}\sum_{j=0}^{\infty}\frac{(-1)^k (z/3)^{k+i+j}}{(\alpha_2 j+\beta_2)}$$

Now replacing i-k and j-k by m and n respectively, if we rearrange the resulting triple series (which can be justified by absolute convergence of the series involved), if follows that

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x^m y^n\left(\frac{z}{3}\right)^{m+n}\sum_{k=0}^{\infty}\frac{(-1)^k (z/3)^{3k}}{\Gamma(\alpha_1(m+k)+\beta_1)\Gamma(\alpha_2(n+k)+\beta_2)\Gamma(\alpha_3 k+\beta_3)}.$$

Thus the result (2.1) is proved.

A generalization of generating relation (2.1) can be obtained in the following from

$$\prod_{j=1}^{n}\left(E_{\alpha_1,\beta_1}\left(\frac{zx_j}{n+1}\right)\right).E_{\alpha,\beta}\left(\frac{-z/n+1}{\prod_{j=1}^{n}(x_j)}\right)$$

$$=\sum_{m_1..m_n=-\infty}^{\infty} x_1^{m_1}...x_n^{m_n}\ {}_{\beta_1,\ldots,\beta_n,\beta}^{\alpha_1,\ldots,\alpha_n,\alpha}J_{m_1,\ldots,m_n}(z). \tag{2.3}$$

where ${}_{\beta_1,\ldots,\beta_n,\beta}^{\alpha_1,\ldots,\alpha_n,\alpha}J_{m_1,\ldots,m_n}(z)$

$$=\left(\frac{z}{n+1}\right)^{\sum_{k=1} m_1}\sum_{k=0}^{\infty}\frac{(-1)^k(z^3)^k}{\Gamma(\alpha_1(m_1+k)+\beta_1)\ldots\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha k+\beta)}, \tag{2.4}$$

provided that both member of equation (2.3) exist.

In a similar way, we obtain

Result-II

$$E_{\alpha_1,\beta_1}\left(\frac{zx}{3}\right)E_{\alpha_2,\beta_2}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}\left(\frac{z}{3xy}\right)=\sum_{m,n=-\infty}^{\infty}x^m y^n\ {}_{\beta_1,\beta_2,\beta_3}^{\alpha_1,\alpha_2,\alpha_3}I_{m,n}(z) \tag{2.5}$$

where ${}_{\beta_1,\beta_2,\beta_3}^{\alpha_1,\alpha_2,\alpha_3}I_{m,n}(z)$

$$=\left(\frac{z}{3}\right)^{m+n}\sum_{k=0}^{\infty}\frac{(z/3)^{3k}}{\Gamma(\alpha_1(m+k)+\beta_1)\Gamma(\alpha_2(n+k)+\beta_2)\Gamma(\alpha_3 k+\beta_3)}, \tag{2.6}$$

and thus, a generalization of equation (2.5) can be obtained as follows :

$$\prod_{j=1}^{n}\left(E_{\alpha_j,\beta_j}\left(\frac{zx_j}{n+1}\right)\right).E_{\alpha,\beta}\left(\frac{z/n+1}{\prod_{j=1}^{n}(x_j)}\right)$$

$$= \sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1}\ldots x_n^{m_n} \, {}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}J_{m_1,\ldots,m_n}(z), \tag{2.7}$$

where ${}^{\alpha_1,\alpha_2,\alpha}_{\beta_1,\beta_2,\beta}I_{m_1,\ldots,m_n}(z)$

$$= \left(\frac{z}{n+1}\right)^{\sum_{j=1}^{n} m_j} \sum_{k=0}^{\infty} \frac{[(z/n+1)^{n+1}]^k}{\Gamma(\alpha_1(m_1+k)+\beta_1)\ldots\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha k+\beta)}, \tag{2.8}$$

provided that both sides of equations (2.5) and (2.7) exist.

The proof of result-**II** is similar to result-**I**.

Result-III

The above results may be extended for the entire function (1.6)

$$\prod_{j=1}^{n} \phi(\alpha_j,\beta_1;zx_j)\,\phi\left(\alpha,\beta;z\Big/\prod_{j=1}^{n}(x_j)\right)$$

$$= \sum_{m_1,\ldots,m_n=-\infty}^{\infty} \frac{x_1^{m_1}\ldots x_n^{m_n}}{m_i!\ldots m_n!} \, {}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}F_{m_1,\ldots,m_n} \tag{2.9}$$

where

$${}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}F_{m_1,\ldots,m_n}(z)$$

$$= (z)^{\sum_{j=1}^{b} m_j} \sum_{k=0}^{\infty} \frac{[-(z)^{n+1}]^k}{(m_1+1)_k\ldots(m_n+1)_k\,\Gamma(\alpha_1(m_1+k)+\beta_1)\ldots\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha k+\beta)k!},$$

provided that both sides of equation (2.9) exist. The proof of results (2.9) is similar to the result (2.1).

3. Special Cases :

For $\alpha_i=\beta_i=\alpha=\beta=1$, {i=1,2,…,n}, equations (2.3) and (2.7) reduce to well known generating function [2] of Hyper-Bessel function $J_{m_1,\ldots,m_n}(z)$ of order n and its modified case $I_{m_1,\ldots,m_n}(z)$

$$\exp\left[\frac{z}{n+1}\left(x_1+\ldots+x_n-\frac{1}{x_1\ldots x_n}\right)\right]=\sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1}\ldots x_n^{m_n} J_{m_1,\ldots,m_n}(z) \tag{3.1}$$

where $J_{m_1,\ldots,m_n}(z)={}^{1,\ldots,1,1}_{1,\ldots,1,1}J_{m_1,\ldots,m_n}(z)$

$$=\frac{(z/n+1)^{\sum_{j=1}^{n} m_j}}{(m_1)!\ldots(m_n)!}\,{}_0F_n\left[-;m_1+1,\ldots,m_n+1;-\left(\frac{z}{n+1}\right)^{n+1}\right] \tag{3.2}$$

and

$$\exp\left[\frac{z}{n+1}\left(x_1+\ldots+x_n+\frac{1}{x_1\ldots x_2}\right)\right]=\sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1}\ldots x_n^{m_n} I_{m_1,\ldots,m_n}(z) \tag{3.3}$$

where $I_{m_1,\ldots,m_n}(z)={}^{1,\ldots,1,1}_{1,\ldots,1,1}\,I_{m_1,\ldots,m_n}(z)$

$$=\frac{(z/n+1)^{\sum_{j=1}^{n} m_j}}{(m_1)!\ldots(m_n)!}\,{}_0F_n\left[-;m_1+1,\ldots,m_n+1;\left(\frac{z}{n+1}\right)^{n+1}\right]$$

respectively. (3.4)

For n=2, equations (2.1) and (2.5) give equation (1.9), respectively.

Replacing $\frac{z}{n+1} \to z^2$ and $x_i \to x_i^2$, i={1,2,…,n}, respectively, in equations (2.3) and (2.7), taking α_i=2 and β_i=1, i={1,2,…,n}, and using the relation (1.5), we get

$$\cosh(zx_1)\ldots\cosh(zx_n).\cos\left(\frac{z}{x_1\ldots x_n}\right)$$

$$= \sum_{m_1,\ldots,m_n=-\infty}^{\infty} \frac{x_1^{m_1}\ldots x_n^{m_n}}{(2m_1)!\ldots(2m_n)!} \, {}_0F_{2n+1}\left[-; m_1+\frac{1}{2},\ldots,m_n+\frac{1}{2}, m_1+1,\ldots,m_n+1,\frac{1}{2}; \left(\frac{-z}{4}\right)^{n+1}\right] \tag{3.5}$$

and

$$\cosh(zx_1)\ldots\cosh(zx_n).\cosh\left(\frac{z}{x_1\ldots x_n}\right)$$

$$= \sum_{m_1,\ldots,m_n=-\infty}^{\infty} \frac{x_1^{m_1}\ldots x_n^{m_n}}{(2m_1)!\ldots(2m_n)!} \, {}_0F_{2n+1}\left[-; m_1+\frac{1}{2},\ldots,m_n+\frac{1}{2}, m_1+1,\ldots,m_n+1,\frac{1}{2}; \left(\frac{z}{4}\right)^{n+1}\right] \tag{3.6}$$

On setting $\alpha_i = \alpha = 1/2$ and $\beta_i = \beta = 1$ in (2.7), replacing $\frac{z}{n+1} \to \sqrt{z}$, $x_i \to \sqrt{x_i}$ {i=1,2,…,n}, respectively, and using the relation (1.5), we get

$$\left(\frac{2}{\sqrt{\pi}}\right)^{n+1} \exp\left[-\left(zx_1+\ldots+zx_n+\frac{z}{x_1\ldots x_n}\right)\right]$$

$$Erfc\left(-\sqrt{zx_1}\right)...Erfc\left(-\sqrt{zx_n}\right)Erfc\left(-\sqrt{\frac{z}{x_1...x_n}}\right)$$

$$=\sum_{m_1,...,m_n=-\infty}^{\infty}(\sqrt{x_1})^{m_1}...(\sqrt{x_n})^{m_n}(\sqrt{z})^{\sum_{j=1}^{n}m_j}\sum_{k=0}^{\infty}\frac{\left[\left(\sqrt{z}\right)^{n+1}\right]^k}{\left(\frac{m_1+k}{2}\right)!...\left(\frac{m_n+k}{2}\right)!\left(\frac{k}{2}\right)!},$$

(3.7)

Now putting $\alpha_i=\alpha=1$ and $\beta_i=\lambda_i+1$ {i=1,2,…,n} and β=λ+1 and replacing z/n+1 by $-z^2/4$ and x_i by $x_i{}^2$ {i=1,2,…,n}, respectively, in (2.9) and using the relation (1.7), we get a multiple generating relation involving Bessel Function :

$$J_{\lambda_1}(zx_1)...J_{\lambda_n}(zx_n)J_{\lambda}\left(\frac{z}{x_1...x_n}\right)=\left(\frac{zx_1}{2}\right)^{\lambda_1}...\left(\frac{zx_n}{2}\right)^{\lambda_n}\left(z\Big/2\prod_{j=1}^{n}x_j\right)^{\lambda}$$

$$\sum_{m_1,...,m_n=-\infty}^{\infty}\frac{x_1^{2m_1}...x_n^{2m_n}(-z^2)^{(m_1+...+m_n)}}{(m_1)!...(m_n)!4^{m_1+...+m_n}\prod_{j=1}^{n}(\Gamma(\lambda_j+m_j+1)\Gamma(\lambda+1)}$$

$${}_0F_{2n+1}\left[-;m_1+1,...,m_n+1,\lambda_1+m_1+1,...,\lambda_n+m_n+1;\lambda+1;\left(\frac{-z^2}{4}\right)^{n+1}\right].$$

(3.8)

For n=2, equation (3.9) reduces to

$$J_{\lambda_1}(zx_1)J_{\lambda_2}(zx_2)J_{\lambda}\left(\frac{z}{x_1x_2}\right)=\left(\frac{zx_1}{2}\right)^{\lambda_1}\left(\frac{zx_2}{2}\right)^{\lambda_2}\left(\frac{z}{2x_1x_2}\right)^{\lambda}$$

$$\sum_{m_1,m_2=-\infty}^{\infty}\frac{x_1^{2m_1}x_2^{2m_2}(-z^2/4)^{m_1+m_2}}{(m_1)!(m_2)!\Gamma(\lambda_1+m_1+1)\Gamma(\lambda_2+m_2+1)\Gamma(\lambda+1)}$$

$$ {}_0F_5\left[-;m_1+1,m_2+1,\lambda_1+m_1+1,\lambda_2+m_2+1;\lambda+1;\frac{z^6}{64}\right]. \tag{3.9}$$

Acknowledgement

We are thankful to Professor M.A. Pathan of Aligarh Musilm Umiversity, Aligarh, India for his valuable suggestions.

Reference

1. Agarwal, R.P., A proposd'ume note de M. Pierre Humbert, C.T. Acad. Sci., Paris 236 (1953), 2031-2032.
2. Delerue, P., Ser la calcul symbolique a n variables et Sur les functions hyper Besseliennes I, II, Ann. Soc. Sci. Bruxelles, (1) 67 (1953), 83-104.
3. Erdelyi, A. et.al., Higher transcendental functions, Vol. 3, McGraw Hill Book Co. Inc., New York, 1953.
4. Humbert, P., Le Calcul symbolieuq a variable, Ann. Soc. Sci. Bruxelles, Ser. A., 56 (1936), 26-43.
5. Mittag-Leffler, G.M. Surla representation analytique d'ume branche uniforme d'ume function monogene. Acta. Math., 29 (1905), 101-182.
6. Srivastava, H.M. and Manocha, H.L., A treatise on generating functions, Ellis Horwood, Ltd., Chichester, New York, 1984.
7. Wilman, A., Uber den fundamentalsatz in der teorie der funktiomer $E_\alpha(x)$, Acta. Math., 29 (1905), 191-207 and 217-234.
8. Wright, E.M., The asymptotic expansion of the generalized Bessel function, Proc. Lond. Math. Soc., (2) 38 (1934a), 257-270.