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TWO VARIABLE MODIFIED LAGUERRE POLYNOMIALS AND THEIR GENERATING RELATIONS

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Abstract: In this paper, the notion of two variable modified Laguerre polynomials (TVMLP) $L_{\alpha,\beta,m,n}(x,y)$ is given and their generating relations are derived. The process involves the problem of framing TVMLP into the context of the representation $\uparrow\omega,\mu$ of a Lie algebra $G(0,1)$. Certain new and known generating relations for the polynomials related to TVMLP are also obtained as special cases.

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1. Introduction:

In recent years, a great deal of attention seems to have been paid to a slight variant of the associated Laguerre polynomials (ALP) $L_n^{(p)}(x)$. These so–called modified Laguerre polynomials (MLP) $L_{\alpha,\beta,m,n}(x)$ were introduced by Goyal [4] in the form

$$L_{\alpha,\beta,m,n}(x) = \frac{\beta^n (m)_n}{n!}\ {}_1F_1\left[-n;m;\frac{\alpha x}{\beta}\right], \quad (\beta\neq 0;m\neq 0,-1,-2\ldots). \tag{1.1}$$

We note that

$$L_{\alpha,\beta,m,n}(x) = \beta^n L_n^{(m-1)}\left(\frac{\alpha x}{\beta}\right),$$

which indicate that although the so–called MLP $L_{\alpha,\beta,m,n}(x)$ can be derived from ALP $L_n^{(p)}(x)$, but these MLP are much easier to handle and also more practical in numerical computations.

We define the two variable modified Laguerre polynomials (TVMLP) as follows

$$L_{\alpha,\beta,m,n}(x,y) = \frac{(m)_n (\beta y)^n}{n!}\, {}_1F_1\left[-n; m; \frac{\alpha x}{\beta y}\right]. \tag{1.2}$$

The generating function for TVMLP $L_{\alpha,\beta,m,n}(x,y)$ is given by

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x,y)t^n = (1-\beta ty)^{-m} \exp\left(\frac{-\alpha xt}{(1-\beta ty)}\right) \tag{1.3}$$

These polynomials satisfy the following differential and pure recurrence relations

$$\frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x,y) = \frac{1}{x}\left(\beta y(1-m-n)\, L_{\alpha,\beta,m,n-1}(x,y) + nL_{\alpha,\beta,m,n}(x,y)\right),$$

$$\frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x,y) = \frac{1}{\beta xy}\left((n+1)\, L_{\alpha,\beta,m,n+1}(x,y) + (\alpha x-\beta y(m+n))\, L_{\alpha,\beta,m,n}(x,y)\right),$$

$$\frac{\partial}{\partial y} L_{\alpha,\beta,m,n}(x,y) = \beta(m+n-1)\, L_{\alpha,\beta,m,n-1}(x,y),$$

$$(n+1)\, L_{\alpha,\beta,m,n}(x,y) + (\alpha x-\beta y(m+2n))\, L_{\alpha,\beta,m,n}(x,y) + \beta^2 y^2(m+n-1)\, L_{\alpha,\beta,m,n-1}(x,y) = 0. \tag{1.4}$$

The differential equation satisfied by $L_{\alpha,\beta,m,n}(x,y)$ is

$$\left(x\frac{d^2}{dx^2} + (m-\frac{\alpha x}{\beta y})\frac{d}{dx} + \frac{\alpha n}{\beta y}\right) L_{\alpha,\beta,m,n}(x,y) = 0. \tag{1.5}$$

The TVMLP are linked to the MLP by the following relation

$$L_{\alpha,\beta,m,n}(x,y) = y^n \, L_{\alpha,\beta,m,n}\left(\frac{x}{y}\right). \tag{1.6}$$

We note that for $m=p+1$ and $\alpha=\beta=1$, equation (1.3) reduces to

$$\sum_{n=0}^{\infty} L_n^{(p)}(x,y)t^n = (1-yt)^{-p-1}\exp\left(\frac{-xt}{(1-yt)}\right), \tag{1.7}$$

where $L_n^{(p)}(x,y)$ are the two variable associated Laguerre polynomials (TVALP) ([1];p. 113 (15)).

Further for $p=0$, equation (1.7) becomes

$$\sum_{n=0}^{\infty} L_n(x,y)t^n = (1-ty)^{-1}\exp\left(\frac{-xt}{(1-yt)}\right), \tag{1.8}$$

where $L_n(x,y)$ are the two variable Laguerre polynomials (TVLP) introduced and discussed by Dattoli and Torre [2,3].

Recently Khan and Yasmin [5] obtained Lie–theoretic generating relations for TVLP $L_n(x,y)$ by constructing a three dimensional Lie algebra isomorphic to special linear algebra $sl(2)$ ([6];p.7). In this paper, we derive generating relations involving TVMLP $L_{\alpha,\beta,,m,n}(x,y)$ by using a representation $\uparrow_{\omega,\mu}$ of a four dimensional Lie algebra $G(0,1)$. Certain new generating relations for TVALP $L_n^{(p)}(x,y)$ and TVLP $L_n(x,y)$ are obtained as special cases. Some known generating relations involving MLP $L_{\alpha,\beta,,m,n}(x)$ and ALP $L_n^{(p)}(x)$ also follow as special cases of our main result.

2. Representation $\uparrow_{\omega,\mu}$ and Generating Relations:

We have the following isomorphism ([6];p.36)

$G(0,1) \cong L[G(0,1)]$,

where $L[G(0,1)]$ is the Lie algebra of a complex four dimensional Lie group $G(0,1)$, a multiplicative matrix group with elements ([6];p.9)

$$g(a,b,c,\tau)=\begin{pmatrix}1\, ce^{\tau}\, a\, \tau\\ 0\, e^{\tau}\ b\ 0\\ 0\ 0\ 1\ 0\\ 0\ 0\ 0\ 1\end{pmatrix},\qquad a,b,c,\tau\ \in\ C \tag{2.1}$$

The group $G(0,1)$ is called the complex harmonic oscillator group ([7]; Chapter-10). A basis for $L[G(0,1)]$ is provided by the matrices ([6];p.9).

$$J^{+}=\begin{pmatrix}0000\\0010\\0000\\0000\end{pmatrix},\qquad J^{-}=\begin{pmatrix}0100\\0000\\0000\\0000\end{pmatrix},\quad J^{3}=\begin{pmatrix}0001\\0100\\0000\\0000\end{pmatrix},\qquad \varepsilon=\begin{pmatrix}0010\\0000\\0000\\0000\end{pmatrix}, \tag{2.2}$$

with commutation relations

$$[J^3,J^{\pm}] = \pm J^{\pm},\ [J^+,\ J^-] = -\varepsilon,\ [\varepsilon,J^{\pm}] = [\varepsilon,J^{\pm}] = \theta \tag{2.3}$$

The machinery constructed in ([6]; Chapters 1,2 and 4) will be applied to find a realization of the irreducible representation $\uparrow\omega,\mu$ of $G(0,1)$, where $\omega,\mu\in\ \not\subset$ such that $\mu\neq 0$. The spectrum S of $\uparrow\omega,\mu$ is the set $S = \{-\omega+k;\ k$ a nonnegative integer$\}$.

In particular, we are looking for the functions $f_n(x,y;t) = Z_n(x,y)t^n$ such that

$$J^3 f_n = nf_n,\ Ef_n = \mu f_n,$$

$$J^+ f_n = \mu fn{+}1,\ J^- fn = (n+\omega)\ f_{n-1},$$

$$C_{0,1}\, f_n = (J^+J^- - EJ^3)\ fn = \mu\omega f_n, \tag{2.4}$$

for all $n\in S$. The commutation relations satisfied by the operators $J^{\pm}, J^3$, E are

$$[J^3,J^{\pm}] = \pm J^{\pm},\ [J^{\pm},J^-] = -E,\ [J^{\pm},E] = [J^3,E] = 0. \tag{2.5}$$

The number of possible solutions of equation (2.5) is tremendous. We assume that these operators take the form

$$J^3 = t\frac{\partial}{\partial t}, \qquad\qquad J^+ = \beta y^2 t\frac{\partial}{\partial x} - \alpha yt\ ,$$

$$J- = -\frac{x}{\alpha yt}\frac{\partial}{\partial x} - \frac{1}{\alpha y}\frac{\partial}{\partial t} + \frac{(m-1)}{\alpha yt}, \quad E=1, \tag{2.6}$$

and note that these operators satisfy the commutation relations (2.5).

Following Miller ([6]; Sections 4–6), we can assume ω=0 and μ=1 without any loss of generality for special function theory. In terms of the functions $Z_n(x,y)$, relations (2.4) become

$$\left(\beta y^2\frac{\partial}{\partial x} - \alpha y\right)Z_n(x,y) = Z_{n+1}(x,y),$$

$$\left(\frac{-x}{\alpha y}\frac{\partial}{\partial x} + \frac{m-n-1}{\alpha y}\right)Zn\,(x,y) = nZn_{-1}\,(x,y),$$

$$\left(x\frac{\partial^2}{\partial x^2} - \left(\frac{\alpha x}{\beta y} + (m-n-2)\right)\frac{\partial}{\partial x} + \frac{\alpha(m-1)}{\beta y}\right)Z_n(x,y) = 0,$$

$$n=0,1,2,\ldots.. \tag{2.7}$$

We see from (2.7) that

$$Z_n(x,y) = n!\left(\frac{x}{y}\right)^{m-n-1} L_{\alpha,\beta,m-n,n}(x,y).$$

The functions $f_n(x,y;t) = n!\left(\frac{x}{y}\right)^{m-n-1} L_{\alpha,\beta,m-n,n}(x,y)t^n$, $n\in S$, form a basis for a realization of the representation $\uparrow_{0,1}$ of $G(0,1)$. This realization of $G(0,1)$ can be extended to a local multiplier representation $T(g)$, $g\in G(0,1)$ defined on F, the space of all functions analytic in a neighbourhood of the point $(x^0,y^0,t^0) = (1,1,1)$.

Following Miller ([6]; p. 18, Theorem 1.10) and using operators (2.6), the local multiplier representation takes the form

$$[T(\exp \tau J^3)f]\,(x,y;t) = f(x,y;te^{\tau}),$$

$$[T(\exp bJ^+)f](x,y;t) = \exp(-\alpha byt)f((x+\beta by^2t), y;t),$$

$$[T\ (\exp cJ^-)f](x,y;t) = \left(1-\frac{c}{\alpha yt}\right)^{1-m} f\left(x\left(1-\frac{c}{\alpha yt}\right), y;t\left(1-\frac{c}{\alpha yt}\right)\right),$$

$$[T\ (\exp a\varepsilon)\ f](x,y;t) = \exp(a)f(x,y;t), \tag{2.8}$$

for $f \in F$. If $g \in G(0,1)$ has parameters a,b,c,τ, then

$T(g)f = T(\exp a\varepsilon)\ T(\exp bJ^+)\ T(\exp cJ^-)\ T(\exp \tau J^3)$ and therefore we obtain

$$[T(g)f](x,y;t) = \left(1-\frac{c}{\alpha yt}\right)^{1-m} \exp\,(a-\alpha byt)$$

$$f\left((x+\beta by^2t)\left(1-\frac{c}{\alpha yt}\right), y; te^{\tau}\left(1-\frac{c}{\alpha yt}\right)\right). \tag{2.9}$$

The matrix elements of $T(g)$ with respect to the analytic basis $f_n(x,y;t) = n!\left(\frac{x}{y}\right)^{m-n-1} L_{\alpha,\beta,m-n,n}(x,y)t^n$, are the function $A_{lk}(g)$, uniquely determined by $\uparrow\omega,\mu$ of $G(0,1)$, and we obtain relations

$$[T(g)f_k](x,y;t) = \sum_{l=0}^{\infty} A_{lk}\ \ (g), f_l(x,y;t), \qquad k = 0,1,2.... \tag{2.10}$$

which simplifies to the identity

$$\exp(a+k\tau-\alpha byt)\left(1+\frac{\beta by^2t}{x}\right)^{m-k-1} k!L_{\alpha,\beta,m-k,k}\left((x+\beta by^2t)\left(1-\frac{c}{\alpha yt}\right), y\right)$$

$$= \sum_{l=0}^{\infty} A_{lk}(\mathrm{g})l!\left(\frac{x}{y}\right)^{k-l} L_{\alpha,\beta,m-l,l}(x,y)\ t^{l-k}, \quad k=0,1,2., \tag{2.11}$$

and the matrix elements $A_{lk}(g)$ are given by ([6]; p. 87 (4.26)).

$$A_{lk}(g) = \exp(a+k\tau)\, c^{k-l}\, L_l^{(k-l)}(-bc), \qquad k,l \geq 0 \tag{2.12}$$

Substituting (2.12) into (2.11) and simplifying, we obtain the generating relation

$$\exp(-\alpha byt)\left(1+\frac{\beta by^2t}{x}\right)^{m-k-1} k!\; L_{\alpha,\beta,m-k,k}\left((x+\beta by^2t)\left(1-\frac{c}{\alpha yt}\right),y\right)$$

$$= \sum_{l=0}^{\infty} c^{k-l}\, L_l^{(k-l)}(-bc)\; l!\; \left(\frac{x}{y}\right)^{k-l} L_{\alpha,\beta,m-l,l}(x,y)\, t^{l-k},$$

$$k=0,1,2\ldots. \tag{2.13}$$

3. Special Cases:

We consider certain special cases of (2.13).

I. Taking b=0 in (2.13) and making use of the limit. ([6]; p. 88 (4.29)), we obtain

$$L_{\alpha,\beta,m-k,k}\left(x\left(1-\frac{c}{\alpha yt}\right),y\right) = \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{cx}{yt}\right)^l L_{\alpha,\beta,m-k+l,k-l}(x,y). \tag{3.1}$$

Further taking $\alpha=\beta=1$ and replacing c by b, x by t and m by p+1 in (3.1), we get

$$L_k^{(p-k)}\left(\left(x-\frac{b}{y}\right),y\right) = \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{c}{y}\right)^l L_{k-l}^{(p-k+l)}(x,y), \tag{3.2}$$

which for y=1, reduces to ([10]; p. 342 (22)) and also to ([6]; p. 113, for $p-k=q$).

II. Taking c=0 in (2.13) and making use of the limit. ([6]; p. 88 (4.29)), we obtain

$$\exp(-\alpha byt)\left(1+\frac{\beta by^2t}{x}\right)^{m-k-1} L_{\alpha,\beta,m-k,k}\left(x\left(1+\frac{\beta by^2\, t}{x}\right),y\right)$$

$$= \sum_{l=0}^{\infty} \binom{l+k}{l} \left(\frac{byt}{x}\right)^{l} L_{\alpha,\beta,m-l-k,l+k}(x,y). \tag{3.3}$$

Further, taking $\alpha=\beta=1$ and replacing x by t, m by $p+1$ in (3.3), we get

$$\exp(-byt)\,(1+by^2)p-k\; L_k^{(p-k)}\left(x\left(1+by^2\right),y\right)$$

$$= \sum_{l=0}^{\infty} \binom{l+k}{l} (by)^l\, L_{l+k}^{(p-l-k)}(x,y). \tag{3.4}$$

Taking $y=1$ and replacing $(p-k)$ by q in (3.4), we get

$$\exp(-bt)\,(1+b)^q\, L_k^{(q)}\left(x(1+b)\right) = \sum_{l=0}^{\infty} b^l \binom{l+k}{l} L_{l+k}^{(q-l)}(x), \tag{3.5}$$

which is the correct form of Miller's result ([6]; p. 112)

Taking $\alpha=\beta=1$ and replacing m by $p+1$ in (2.13), we get the

following generating relation for TVALP

$$\exp(-byt)\left(1+\frac{by^2t}{x}\right)^{p-k} k!\; L_k^{(p-k)}\left(x\left(1+\frac{by^2t}{x}\right)\left(1-\frac{c}{yt}\right),y\right)$$

$$= \sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc)\left(\frac{x}{y}\right)^{k-l} l!\, L_l^{(p-l)}(x,y)\, t^{l-k},$$

$$\left|\frac{by^2t}{x}\right| < 1,\; k=0,1,2\ldots, \tag{3.6}$$

which for $y=1$, gives a result of Miller ([6]; p. 112 (4.94)).

Further, replacing p and k by l in (3.6), we obtain the following generating relation for TVLP

$$\exp(-byt)\, L_l\left((x+by^2t)\left(1-\frac{c}{yt}\right),y\right) = \sum_{l=0}^{\infty} L_l(-bc)\, L_l(x,y). \tag{3.7}$$

IV. Taking y=1 in (2.13), we get the following generating relation for MLP

$$exp(-\alpha bt)\left(1+\frac{\beta bt}{x}\right)^{m-k-1} k!\, L_{\alpha,\beta,m-k,k}\left((x+\beta bt)\left(1-\frac{c}{\alpha t}\right)\right)$$

$$=\sum_{l=0}^{\infty} c^{k-l}\, L_l^{(k-l)}(-bc)\, l!\; x^{k-l}\, L_{\alpha,\beta,m-l,l}(x)\, t^{l-k}, \quad k=0,1,2,\ldots, \tag{3.8}$$

considered by Pathan and Khan [9].

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