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ON REDUCIBILITY OF CERTAIN q-DOUBLE HYPERGEOMETRIC SERIES AND CLAUSEN TYPE IDENTITIES

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Abstract: In this paper, a method has been developed to establish certain transformations of double q-series in terms of basic hypergeometric functions of one variable. These results lead to certain interesting Clausen type identities. We also discuss certain continued fraction representations involving q-series.

Key words & Phrases: q-series, basic hypergeometric series, Clausen identities.

AMS Classification: 33A30.

1. Introduction:

In this paper, we have made use of certain known summations to establish transformations of q-double series in terms of a single series. We have deduced Clausen type identities from these results. We also discuss some interesting continued fraction representations involving q-series.

For α, real or complex and $|q|<1$, we define the q-shifted factorials by

$$[\alpha;q]_n = \begin{cases} 1 & if\ n=0 \\ (1-\alpha)(1-\alpha q)\ldots(1-\alpha q^{n-1}), & if\ n=1,2,3,\ldots \end{cases} \tag{1}$$

A basic hypergeometric function is defined as :

$$ {}_r\Phi_s\left[\begin{matrix} a_1,a_2,...,a_r;q;z \\ b_1,b_2,...,b_s;q^{\lambda}\end{matrix}\right] $$

$$ =\sum_{n=0}^{\infty}\frac{[a_1,a_2,...,a_r;q]_n z^n q^{\lambda n(n-1)/2}}{[q,b_1,b_2,...,b_s;q]_n}, \tag{2} $$

where $[a_1,a_2,...a_r;q]_n = [a_1;q]_n [a_2;q]_n ...[a_r;q]_n$.

The series ${}_r\phi_s$ converges absolutely for all z if $\lambda>0$ and for $|z|<1$ if $\lambda=0$. Other notations and definitions appearing in this paper have their usual meaning. We shall use the following series identity to establish our results,

$$ \sum_{n=0}^{\infty}\sum_{k=0}^{n}B(n,k)=\sum_{n,k=0}^{\infty}B(n+k,k), \tag{3} $$

provided the series on both sides of (1.3) exist.

We shall use the following known summations of q-series in our analysis :

$$ {}_2\Phi_1\left[\begin{matrix} q^{-n},a;q;cq^n/a \\ c\end{matrix}\right]=\frac{[c/a;q]_n}{[c;q]_n}. \tag{4} $$

[4; App.II (II.7)]

$$ {}_3\Phi_2\left[\begin{matrix} a,b,q^{-n} \quad ;q;q \\ c,abq^{1-n}/c\end{matrix}\right]=\frac{[c/a,c/b;q]_n}{[c,c/ab;q]_n}. \tag{5} $$

[4; App.II (II.12)]

$$ {}_2\Phi_1\left[\begin{matrix} x,q^{-n} \quad ;q;-q/x \\ q^{-n}/x\end{matrix}\right]=\frac{[q;q]_n[x^2q^2;q^2]_m}{[xq;q]_n[q^2;q^2]_m}, \tag{6} $$

[5; (2.8) p. 1024]

where m is the greatest integer $\leq$ n/2.

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n},-q^{-n}/xy,x,y \quad ;q;q \\ -xyq,q^{-n}/x,q^{-n}/y\end{matrix}\right]=\frac{[q,xyq;q]_n[x^2q^2,y^2q^2;q^2]_m}{[xq,yq;q]_n[q^2,x^2y^2q^2;q^2]_m}, \tag{7} $$

where m is the greatest integer ≤ n/2.

[5; (2.5) p. 1024]

$$ {}_2\Phi_1\left[\begin{matrix} x, q^{-n} \quad ; q; -1/x \\ q^{-n}/x \end{matrix}\right] = \frac{[q;q]_n [x^2q^2;q^2]_m x^{n-2m}}{[xq;q]_n [q^2;q^2]_m}, \tag{8} $$

[5; (2.9) p. 1024]

where m is the greatest integer ≤ n/2.

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, -q^{-n}/xy, xq, yq \quad ; q; q \\ -xyq, q^{1-n}/x, q^{1-n}/y \end{matrix}\right] $$

$$ = \frac{(-)^n [q;q]_n [xyq;q]_n [x^2q^2, y^2q^2; q^2]_m}{q^n [x, y; q]_n [q^2, x^2y^2q^2; q^2]_m}, \tag{9} $$

where m is the greatest integer ≤ n/2.

[5; (2.10) p.1025]

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, -q^n/x^2, y, -y; q; q \\ q^{-n}/x, -q^{-n}/x, y^2q \end{matrix}\right] $$

$$ = \frac{[q;q]_n [x^2y^2q^2;q^2]_n [x^2q^2, y^2q^2; q^2]_m}{[x^2q^2;q^2]_n [y^2q;q]_n [q^2, x^2y^2q^2; q^2]_m}, \tag{10} $$

where m is the greatest integer ≤ n/2.

[5; (2.12) p. 1025]

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, o; q; q \\ q^{-n}/x, -q^{-n}/x \end{matrix}\right] = \frac{[q;q]_n [x^2q^2;q^2]_m}{[x^2y^2;q^2]_n [q^2;q^2]_m}, \tag{11} $$

where m is the greatest integer ≤ n/2.

[5; (2.14) p.1026]

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, o \quad ; q; 1 \\ q^{-n}/x, -q^{-n}/x; q \end{matrix}\right] = \frac{[q;q]_n [x^2q^2;q^2]_m q^{n(n+1)/2} x^{2n-2m}}{[x^2y^2;q^2]_n [q^2;q^2]_m}, \tag{12} $$

where m is the greatest integer ≤ n/2.

[5;(2.15) p.1026]

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, q^{-n}/x^2, yq, -yq ; q ; q \\ q^{1-n}/x, -q^{1-n}/x, y^2q \end{matrix}\right] $$

$$ = \frac{(-)^n [q;q]_n [x^2y^2q^2;q^2]_n [x^2q^2, y^2q^2;q^2]_m}{q^2 [x^2;q^2]_n [y^2q;q]_n [q^2, x^2y^2q^2;q^2]_m}, \tag{13} $$

where m is the greatest integer ≤ n/2.

[5;(2.16) p.1026]

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 0 \ ; q ; q \\ q^{1-n}/x, -q^{1-n}/x \end{matrix}\right] = \frac{(-)^n [q;q]_n [x^2q^2;q^2]_m}{q^n [x^2;q^2]_n [q^2;q^2]_m}, \tag{14} $$

where m is the greatest integer ≤ n/2.

[5;(2.18) p. 1026]

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 0 \quad ; q ; q^2 \\ q^{1-n}/x, -q^{1-n}/x ; q \end{matrix}\right] $$

$$ = \frac{(-)^n [q;q]_n [x^2q^2;q^2]_m q^{n(n-1)/2} x^{2n-2m}}{[x^2;q^2]_n [q^2;q^2]_m}, \tag{15} $$

where m is the greatest integer ≤ n/2.

[5; (2.19) p. 1026]

Putting yq^{-n} for y in [5; (2.20) p.1027] we get the following summation formula:

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 1/xy, -1/xy ; q ; q \\ q^{1-n}/x, -q^{-n}/x, 1/x^2y^2 \end{matrix}\right] $$

$$ = \frac{(-)^n (xq)^{-n} [q;q]_n [1/y^2;q^2]_n [x^2q^2;q^2]_m [y^2q^2;q^2]_{m-n}}{[x, -xq;q]_n [1/x^2y^2;q]_n [q^2;q^2]_m [x^2y^2q^2;q^2]_{m-n}}, \tag{16} $$

where m is the greatest integer ≤ n/2.

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 0 \ ;q;q \\ q^{1-n}/x, -q^{-n}/x \end{matrix}\right] = \frac{(-)^n [q;q]_n [x^2q^2;q^2]_m x^{n-2m}}{[x,-xq;q]_n [q^2;q^2]_m}, \tag{17} $$

where m is the greatest integer ≤ n/2.

[5; (2.23) p. 1027]

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 0 \ ;q;q \\ q^{1-n}/x, -q^{-n}/x;q \end{matrix}\right] = \frac{(-)^n x^n q^{n(n-1)/2} [q;q]_n [x^2q^2;q^2]_m}{[x,-xq;q]_n [q^2;q^2]_m}, \tag{18} $$

where m is the greatest integer ≤ n/2.

[5; (2.24) p.1027]

Putting wq^m for w in [3; (4.3) p. 420] we get the summation formula ;

$$ {}_4\Phi_3\left[\begin{matrix} a, aq, a^2q^{2-2m}/w^2, q^{-2m}; q^2; q^2 \\ a^2q^2, aq^{1-2m}/w, aq^{2-2m}/w \end{matrix}\right] = \frac{[w;q]_{2m} [w/a.-q;q]_m}{[w/a;q]_{2m} [w,-aq;q]_m}. \tag{19} $$

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, c, d, \frac{1}{cd} q^{\frac{1}{2}-n} ;q;q^2 \\ \frac{1}{c}q^{1-n}, \frac{1}{d}q^{1-n}, cdq^{1/2} \end{matrix}\right] = \frac{[cd;q]_n [c,d,-q^{1/2};q^{1/2}]_n}{[c,d;q]_n [cd;q^{1/2}]_n}. \tag{20} $$

[6;(1.4) p. 72]

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, c, d, \frac{1}{cd} q^{\frac{3}{2}-n} \quad ;q;q \\ \frac{1}{c}q^{1-n}, \frac{1}{d}q^{1-n}, cdq^{1/2} \end{matrix}\right] $$

$$ = \frac{[cdq^{-1/2};q^{1/2}]_{2n} [c,d;q^{1/2}]_n [q;q]_n}{[cdq^{-1/2};q^{1/2}]_n [cdq^{1/2};q]_n [c,d;q]_n [q^{1/2};q^{1/2}]_n}. \tag{21} $$

[6; (4.13) p.79]

$$
{}_4\Phi_3\left[\begin{matrix} q^{-n}, c, d, \frac{1}{cd}q^{\frac{1}{2}-n} & ;q;q \\ \frac{1}{c}q^{1-n}, \frac{1}{d}q^{1-n}, cdq^{-1/2} & \end{matrix}\right]
$$

$$
= \frac{[q,cd;q]_n [c,d;q^{1/2}]_n q^{-n/2}}{[c,d;q]_n [cdq^{-1/2};q^{1/2}]_n [q^{1/2};q^{1/2}]_n}. \tag{22}
$$

[6; (4.18) p.80]

2. Main Results :

In this section we shall establish certain transformations of double series in terms of single series.

(i) Multiplying both sides of (1.4) by an arbitrary sequence B_n, summing over n from 0 to ∞, applying the identity (1.3) and then replacing B_n by $\frac{z^n}{[q;q]_n} A_n$, , where A_n is another arbitrary sequence, we get :

$$
\sum_{n,k=0}^{\infty} A_{n+k} \frac{[a;q]_k (-cz/a)^k z^n q^{k(k-1)/2}}{[c;q]_k [q;q]_k [q;q]_n} = \sum_{n=0}^{\infty} A_n \frac{[c/a;q]_n z^n}{[q,c;q]_n}. \tag{1}
$$

This is a transformation which reduces a double series in terms of a single series.

Similarly, one can easily establish the following results:

(ii)
$$
\sum_{n,k=0}^{\infty} A_{n+k} \frac{[a,b;q]_k [c/ab;q]_n (cz/ab)^k z^n}{[q,c;q]_k [q;q]_n} = \sum_{n=0}^{\infty} A_n \frac{[c/a,c/b;q]_n z^n}{[q,c;q]_n}. \tag{2}
$$

(using (1.5) with $B_n = \frac{[c/ab;q]_n z^n}{[q;q]_n} A_n$)

(iii) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x;q]_k [xq;q]_n (-zq)^k z^n}{[q;q]_k [q;q]_n} = \sum_{n=0}^{\infty} A_n \frac{[x^2q^2;q^2]_m z^n}{[q^2;q^2]_m}, \tag{3}$$

where m is the greatest integer $\leq$ n/2.

(using (1.6) with $B_n = \frac{[xq;q]_n z^n}{[q;q]_n} A_n$)

(iv) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x,y;q]_k [xq,yq;q]_n (-zq)^k z^n}{[q,-xyq;q]_k [q,-xyq;q]_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{[xyq;q]_n [x^2q^2, y^2q^2;q^2]_m z^n}{[-xyq;q]_n [q^2, x^2y^2q^2;q^2]_m}, \tag{4}$$

where m is the greatest integer $\leq$ n/2.

(using (1.7) with $B_n = \frac{[xq,yq;q]_n z^n}{[q,-xyq;q]_n} A_n$).

(v) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x;q]_k [xq;q]_n (-z)^k z^n}{[q;q]_k [q;q]_n} = \sum_{n=0}^{\infty} A_n z^n \frac{[x^2q^2;q^2]_m x^{n-2m}}{[q^2;q^2]_m},$$

where m is the greatest integer $\leq$ n/2.

(using (1.8) with $B_n = \frac{[xq;q]_n z^n}{[q;q]_n} A_n$).

(vi) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[xq,yq;q]_k [x,y;q]_n (-z/q)^k z^n}{[q,-xyq;q]_k [q,-xyq;q]_n}$$

$$= \sum_{n=0}^{\infty} A_n \left(\frac{z}{q}\right)^n \frac{(-)^n [xyq;q]_n [x^2q^2, y^2q^2;q^2]_m}{[-xyq;q]_n [q^2, x^2y^2q^2;q^2]_m}, \tag{6}$$

where m is the greatest integer $\leq$ n/2.

(using (1.9) with $B_n = \frac{[x,y;q]_n z^n}{[q,-xyq;q]_n} A_n$)

(vii)
$$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[y,-y;q]_k [xq,-xq;q]_n (-zq)^k z^n}{[q,y^2q;q]_k [q,x^2q;q]_n}$$

$$= \sum_{n=0}^{\infty} A_n z^n \frac{[x^2y^2q^2;q^2]_n [x^2q^2,y^2q^2;q^2]_m}{[x^2q,y^2q;q]_n [q^2,x^2y^2q^2;q^2]_m}, \quad (7)$$

where m is the greatest integer $\le$ n/2.

(using (1.10) with $B_n = \frac{[xq,-xq;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(viii)
$$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[xq,-xq;q]_n (-zq)^k z^n}{[q,x^2q;q]_k [q;q]_k} = \sum_{n=0}^{\infty} A_n z^n \frac{[x^2q^2;q^2]_m}{[x^2q;q]_n [q^2;q^2]_m}, \quad (8)$$

where m is the greatest integer $\le$ n/2.

(using (1.11) with $B_n = \frac{[xq,-xq;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(ix)
$$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[xq,-xq;q]_k (-z)^k z^n q^{k(k-1)/2}}{[q,x^2q;q]_n [q;q]_k}$$

$$= \sum_{n=0}^{\infty} A_n z^n \frac{[x^2q^2;q^2]_m x^{2n-2m} q^{n(n-1)/2}}{[x^2q;q]_n [q^2;q^2]_m}, \quad (9)$$

where m is the greatest integer $\le$ n/2.

(using (1.12) with $B_n = \frac{[xq,-xq;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(x)
$$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[yq,-yq;q]_k [x,-x;q]_n (-z/q)^k z^n}{[q,y^2q;q]_k [q,x^2q;q]_n}$$

$$= \sum_{n=0}^{\infty} A_n z^n \frac{(-)^n [x^2y^2q^2;q^2]_n [x^2q^2, y^2q^2;q^2]_m}{[x^2q, y^2q;q]_n [q^2, x^2y^2q^2;q^2]_m q^n}, \tag{10}$$

where m is the greatest integer ≤ n/2.

(using (1.13) with $B_n = \frac{[x,-x;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(xi)
$$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x,-x;q]_n (-z/q)^k z^n}{[q,x^2q;q]_n [q;q]_k} = \sum_{n=0}^{\infty} A_n \frac{[x^2q^2;q^2]_m (z/q)^n}{[x^2q;q]_n [q^2;q^2]_m}, \tag{11}$$

where m is the greatest integer ≤ n/2.

(using (1.14) with $B_n = \frac{[x,-x;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(xii)
$$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x,-x;q]_n (-z)^k z^n q^{k(k-1)/2}}{[q,x^2q;q]_n [q;q]_k}$$

$$= \sum_{n=0}^{\infty} A_n (-z)^n \frac{[x^2q^2;q^2]_m x^{n-2m} q^{n(n-1)/2}}{[x^2q;q]_n [q^2;q^2]_m}, \tag{12}$$

where m is the greatest integer ≤ n/2.

(using (1.15) with $B_n = \frac{[x,-x;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(xiii)
$$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[1/xy,-1/xy;q]_k [x,-xq;q]_n (-z)^k z^n}{[q,1/x^2y^2;q]_k [q,x^2q;q]_n}$$

$$= \sum_{n=0}^{\infty} A_n (-z/xq)^n \frac{[y^{-2};q^2]_n [x^2q^2;q]_m [y^2q^2;q^2]_{m-n}}{[x^{-2}y^{-2};q]_n [x^2q;q]_n [q^2;q^2]_m [x^2y^2q^2;q^2]_{m-n}}, \tag{13}$$

where m is the greatest integer ≤ n/2.

(using (1.16) with $B_n=\frac{[x,-xq;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(xiv) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x,-xq;q]_n (-z)^k z^n}{[q,x^2q;q]_n [q;q]_k}$$

$$= \sum_{n=0}^{\infty} A_n (-z)^n \frac{[x^2q^2;q^2]_m x^{n-2m}}{[x^2q;q]_n [q^2;q^2]_m}, \quad (14)$$

where m is the greatest integer $\le$ n/2.

(using (1.17) with $B_n=\frac{[x,-xq;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(xv) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x,-xq;q]_n (-z)^k z^n q^{k(k-1)/2}}{[q,x^2q;q]_n [q;q]_k}$$

$$= \sum_{n=0}^{\infty} A_n \frac{(-zx)^n q^{n(n-1)/2} [x^2q^2;q^2]_m}{[x^2q;q]_n [q^2;q^2]_m}, \quad (15)$$

where m is the greatest integer $\le$ n/2.

(using (1.18) with $B_n=\frac{[x,-xq;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(xvi) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[a,aq;q^2]_k [wq/a,w/a;q^2]_n (zq)^k z^n}{[q^2,a^2q^2;q^2]_k [q^2,w^2/a^2;q^2]_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{[w;q]_{2n} z^n}{[w;q]_n [q;q]_n [-aq,-w/a;q]_n}, \quad (16)$$

(using (1.19) with $B_n=\frac{[w/a,wq/a;q^2]_n z^n}{[q^2,w^2/a^2;q^2]_n} A_n$)

(xvii) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[c,d;q]_k [c,d;q]_n (zq^{1/2})^k z^n}{[q,cdq^{1/2};q]_k [q,cdq^{1/2};q]_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{[cd;q]_n [c,d;q^{1/2}]_n [-q^{1/2};q^{1/2}]_n z^n}{[q,cdq^{1/2};q]_n [cd;q^{1/2}]_n}. \tag{17}$$

(using (1.20) with $B_n = \frac{[c,d;q]_n z^n}{[q,cdq^{1/2};q]_n} A_n$)

(xviii) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[c,d;q]_k [c,d;q]_n (zq^{1/2})^k z^n}{[q,cdq^{1/2};q]_k [q,cdq^{-1/2};q]_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{[cdq^{-1/2};q^{1/2}]_n [c,d;q^{1/2}]_n z^n}{[cdq^{-1/2},q^{1/2};q^{1/2}]_n [cdq^{1/2};q]_n}. \tag{18}$$

(using (1.21) with $B_n = \frac{[c,d;q]_n z^n}{[q,cdq^{-1/2};q]_n} A_n$)

(xix) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[c,d;q]_k [c,d;q]_n (zq^{-1/2})^k z^n}{[q,cdq^{-1/2};q]_k [q,cdq^{1/2};q]_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{[cd;q]_n [c,d;q^{1/2}]_n (zq^{-1/2})^n}{[cdq^{1/2};q]_n [cdq^{-1/2},q^{1/2};q^{1/2}]_n}. \tag{19}$$

(using (1.22) with $B_n = \frac{[c,d;q]_n z^n}{[q,cdq^{1/2};q]_n} A_n$)

3. Clausen type Identities:

In this section, we deduce the Clausen type identities from the result established in section 3.

(i) Taking A_n=1 in (2.2), we get :

$$ {}_2\Phi_1\begin{bmatrix} a,b;q;cz/ab \\ c \end{bmatrix} {}_1\Phi_0\begin{bmatrix} c/ab;q;z \\ - \end{bmatrix} = {}_2\Phi_1\begin{bmatrix} c/a,c/b;q;z \\ c \end{bmatrix}, \tag{1} $$

which is the basic analogue of Euler's transformation.

(ii) For A_n=1, (2.4) yields the product formula :

$$ {}_2\Phi_1\begin{bmatrix} x,y;q;-zq \\ -xyq \end{bmatrix} {}_2\Phi_1\begin{bmatrix} xq,yq;q;z \\ -xyq \end{bmatrix} $$

$$ = {}_4\Phi_3\begin{bmatrix} xyq,xyq^2,x^2q^2,y^2q^2;q^2;z^2 \\ -xyq,-xyq^2,x^2y^2q^2 \end{bmatrix} $$

$$ +\frac{z(1-xyq)}{(1+xyq)}\,{}_4\Phi_3\begin{bmatrix} xyq^2,xyq^3,x^2q^2,y^2q^2;q^2;z^2 \\ -xyq^2,-xyq^3,x^2y^2q^2 \end{bmatrix}, \tag{2} $$

which is a known result [5;(2.37) p. 1031].

Similarly, taking A_n=1 in (2.6)-(2.19) we have the following results respectively.

(iii)
$$ {}_2\Phi_1\begin{bmatrix} xq,yq;q;-z/q \\ -xyq \end{bmatrix} {}_2\Phi_1\begin{bmatrix} x,y;q;z \\ -xyq \end{bmatrix} $$

$$ = {}_4\Phi_3\begin{bmatrix} xyq,xyq^2,x^2q^2,y^2q^2;q^2;z^2/q^2 \\ -xyq,-xyq^2,x^2y^2q^2 \end{bmatrix} $$

$$ -\frac{z(1-xyq)}{q(1+xyq)}\,{}_4\Phi_3\begin{bmatrix} xyq^2,xyq^3,x^2q^2,y^2q^2;q^2;z^2/q^2 \\ -xyq^2,-xyq^3,x^2y^2q^2 \end{bmatrix}. \tag{3} $$

(iv)
$$ {}_2\Phi_1\begin{bmatrix} y,-y;q;-zq \\ y^2q \end{bmatrix} {}_2\Phi_1\begin{bmatrix} xq,-xq;q;z \\ x^2q \end{bmatrix} $$

$$ = {}_4\Phi_3\begin{bmatrix} xyq,-xyq,xyq^2,-xyq^2;q^2;z^2 \\ x^2q,y^2q,x^2y^2q^2 \end{bmatrix} $$

$$+\frac{z(1-x^2y^2q^2)}{(1-x^2q)(1-y^2q)}\,{}_4\Phi_3\left[\begin{matrix}xyq^2,-xyq^2,xyq^3,-xyq^3;q^2;z^2\\ x^2q^3,y^2q^3,x^2y^2q^2\end{matrix}\right]. \tag{4}$$

(v) $${}_2\Phi_1\left[\begin{matrix}xq,-xq;q;z\\ x^2q\end{matrix}\right]$$

$$=[-zq;q]_\infty\,{}_0\Phi_1\left[\begin{matrix}-;q^2;z^2\\ x^2q;\end{matrix}\right]+\frac{z[-zq;q]_\infty}{(1-x^2q)}\,{}_0\Phi_1\left[\begin{matrix}-;q^2;z^2\\ x^2q^3;\end{matrix}\right]. \tag{5}$$

(vi) $$[z;q]_\infty\,{}_2\Phi_1\left[\begin{matrix}xq,-xq;q;z\\ x^2q\end{matrix}\right]$$

$$={}_0\Phi_1\left[\begin{matrix}-;q^2;x^2z^2q^3\\ x^2q;q^4\end{matrix}\right]+\frac{x^2zq}{(1-x^2q)}\,{}_0\Phi_1\left[\begin{matrix}-;q^2;x^2z^2q^5\\ x^2q^3;q^4\end{matrix}\right]. \tag{6}$$

(vii) $${}_2\Phi_1\left[\begin{matrix}yq,-yq;q;-z/q\\ y^2q\end{matrix}\right]{}_2\Phi_1\left[\begin{matrix}x,-x;q;z\\ x^2q\end{matrix}\right]$$

$$={}_4\Phi_3\left[\begin{matrix}xyq,-xyq,xyq^2,-xyq^2;q^2;z^2/q^2\\ x^2q,y^2q,x^2y^2q^2\end{matrix}\right]$$

$$-\frac{z(1-x^2y^2q^2)}{(1-x^2q)(1-y^2q)}\,{}_4\Phi_3\left[\begin{matrix}xyq^2,-xyq^2,xyq^3,-xyq^3;q^2;z^2/q^2\\ x^2q^3,y^2q^3,x^2y^2q^2\end{matrix}\right], \tag{7}$$

(viii) $${}_2\Phi_1\left[\begin{matrix}x,-x;q;z\\ x^2q\end{matrix}\right]=[-z/q;q]_\infty\,{}_0\Phi_1\left[\begin{matrix}-;q^2;z^2/q^2\\ x^2q;\end{matrix}\right]$$

$$-\frac{z[-z/q;q]_\infty}{q(1-x^2q)}\,{}_0\Phi_1\left[\begin{matrix}-;q^2;z^2/q^2\\ x^2q^3;\end{matrix}\right]. \tag{8}$$

(ix) $$[z;q]_\infty\,{}_2\Phi_1\left[\begin{matrix}x,-x;q;z\\ x^2q\end{matrix}\right]$$

$$= {}_0\Phi_1\left[\begin{matrix}-;q^2;x^2z^2q\\ x^2q;q^4\end{matrix}\right]-\frac{x^2z}{(1-x^2q)}\,{}_0\Phi_1\left[\begin{matrix}-;q^2;x^2z^2q^3\\ x^2q^3;q^4\end{matrix}\right]. \tag{9}$$

(x) $${}_2\Phi_1\left[\begin{matrix}1/xy,-1/xy;q;-z\\ 1/x^2y^2\end{matrix}\right]{}_2\Phi_1\left[\begin{matrix}x,-xq;q;z\\ x^2q\end{matrix}\right]$$

$$= {}_4\Phi_3\left[\begin{matrix}1/y,-1/y,q/y,-q/y;q^2;z^2/q^2\\ x^2q,1/y^2,q/x^2y^2\end{matrix}\right]$$

$$-\frac{zx}{q(1-x^2q)}\,{}_4\Phi_3\left[\begin{matrix}q/y,-q/y,q^2/y,-q^2/y;q^2;z^2/q^2\\ x^2q^3,q^2/y^2,q/x^2y^2\end{matrix}\right]. \tag{10}$$

(xi) $${}_2\Phi_1\left[\begin{matrix}x,-xq;q;z\\ x^2q\end{matrix}\right]=[-z;q]_\infty\,{}_0\Phi_1\left[\begin{matrix}-;q^2;z^2\\ x^2q;\end{matrix}\right]$$

$$-\frac{xz[-z;q]_\infty}{(1-x^2q)}\,{}_0\Phi_1\left[\begin{matrix}-;q^2;z^2\\ x^2q^3;\end{matrix}\right]. \tag{11}$$

(xii) $$[z;q]_\infty\,{}_2\Phi_1\left[\begin{matrix}x,-xq;q;z\\ x^2q\end{matrix}\right]$$

$$= {}_0\Phi_1\left[\begin{matrix}-;q^2;x^2z^2q\\ x^2q;q^4\end{matrix}\right]-\frac{xz}{(1-x^2q)}\,{}_0\Phi_1\left[\begin{matrix}-;q^2;x^2z^2q^3\\ x^2q^3;q^4\end{matrix}\right]. \tag{12}$$

(xiii) $${}_2\Phi_1\left[\begin{matrix}a,aq;q^2;zq\\ a^2q^2\end{matrix}\right]{}_2\Phi_1\left[\begin{matrix}wq/a,w/a;q^2;z\\ w^2/a^2\end{matrix}\right]$$

$$= {}_4\Phi_3\left[\begin{matrix}\sqrt{w},-\sqrt{w},\sqrt{wq},-\sqrt{wq};q;z\\ w,-w/a,-aq\end{matrix}\right]. \tag{13}$$

(xiv) $${}_2\Phi_1\left[\begin{matrix}c,d\;;q;zq^{1/2}\\ cdq^{1/2}\end{matrix}\right]{}_2\Phi_1\left[\begin{matrix}c,d\;;q;z\\ cdq^{1/2}\end{matrix}\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} c,d,\sqrt{cd},-\sqrt{cd} \\ cd,q^{1/4}\sqrt{cd},-q^{1/4}\sqrt{cd} \end{matrix};q^{1/2};z\right]. \tag{14}$$

(xv) $${}_2\Phi_1\left[\begin{matrix} c,d \\ cdq^{1/2} \end{matrix};q;zq^{1/2}\right]{}_2\Phi_1\left[\begin{matrix} c,d \\ cdq^{-1/2} \end{matrix};q;z\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} c,d,\sqrt{cd},-\sqrt{cd} \\ cdq^{-1/2},q^{1/4}\sqrt{cd},-q^{1/4}\sqrt{cd} \end{matrix};q^{1/2};z\right]. \tag{15}$$

(xvi) $${}_2\Phi_1\left[\begin{matrix} c,d \\ cdq^{-1/2} \end{matrix};q;zq^{-1/2}\right]{}_2\Phi_1\left[\begin{matrix} c,d \\ cdq^{1/2} \end{matrix};q;z\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} c,d,\sqrt{cd},-\sqrt{cd} \\ cdq^{-1/2},q^{1/4}\sqrt{cd},-q^{1/4}\sqrt{cd} \end{matrix};q^{1/2};zq^{-1/2}\right]. \tag{16}$$

4. Certain continued fraction representations.

Continued fraction representation for the ratio of two ${}_2\Phi_1$'s and ${}_3\varphi_2$'s do exist, former with a general argument while the letter with a constant argument of the ϕ-series. There are no direct representation involving j-series ${}_{r+1}\Phi_r$ (r>1) with a general argument. Here we make use of some of the results established in this paper to provide continued fraction representation involving certain ${}_3\varphi_2$ and ${}_4\varphi_3$ with general arguments. Such results provide freedom to compute the results at an arbitrary point inside the region of convergence.

If we put x=1 in (3.2) we have :

$${}_2\Phi_1\left[\begin{matrix} q,yq;q;z \\ -yq \end{matrix}\right] = {}_3\Phi_2\left[\begin{matrix} yq,yq^2,q^2;q^2;z^2 \\ -yq,-yq^2 \end{matrix}\right]$$

$$+\frac{z(1-yq)}{(1+yq)}\,{}_3\Phi_2\left[\begin{matrix} yq^2,yq^3,q^2;q^2;z^2q \\ -yq^2,-yq^3 \end{matrix}\right]. \tag{1}$$

Again, if we take β=1 in [2; (2.24) p. 179] we get :

$$
{}_2\varphi_1\left[\begin{matrix} q,\alpha;q;z \\ \gamma \end{matrix}\right]
$$

$$
=1+\frac{z(1-\alpha)}{o+}\ \frac{(1-\gamma)/(1-\alpha zq)}{1+}\ \frac{z(1-\alpha q)}{o+}\ \frac{(1-\gamma q)/(1-\alpha zq^2)}{1+}\cdots. \tag{2}
$$

Now, making use of (4.1) and (4.2) we have :

$$
{}_3\varphi_2\left[\begin{matrix} yq,yq^2,q^2;q^2;z^2 \\ -yq,-yq^2 \end{matrix}\right]+\frac{z(1-yq)}{(1+yq)}\,{}_3\varphi_2\left[\begin{matrix} yq^2,yq^3,q^2;q^2;z^2 \\ -yq^2,-yq^3 \end{matrix}\right]
$$

$$
=1+\frac{z(1-yq)}{0+}\ \frac{(1+yq)/(1-yzq^2)}{1+}\ \frac{z(1-yq^2)}{0+}\ \frac{(1+yq^2)/(1-yzq^3)}{1+}\cdots \tag{3}
$$

If we take y=1 in (3.4) we get :

$$
{}_2\Phi_1\left[\begin{matrix} xq,-xq;q;z \\ x^2q \end{matrix}\right]={}_4\Phi_3\left[\begin{matrix} xq,-xq,xq^2,-xq^2;q^2;z^2 \\ x^2q,x^2q^2,q \end{matrix}\right]
$$

$$
+\frac{z(1-x^2q^2)}{(1-q)(1-x^2q)}\,{}_4\Phi_3\left[\begin{matrix} xq^2,-xq^2,xq^3,-xq^3;q^2;z^2 \\ x^2q^2,x^2q^3,q^3 \end{matrix}\right]. \tag{4}
$$

Making use of the result [2; (2.19)] we find :

$$
\frac{\displaystyle\sum_{r=0}^{\infty}\frac{[x^2q^2;q^2]_{2r}(z^2q^2)^r}{[x^2q;q]_{2r}[q;q^2]_r}+\frac{zq(1-x^2q^2)}{(1-q)(1-x^2q)}\sum_{r=0}^{\infty}\frac{[x^2q^4;q^2]_{2r}(z^2q^2)^r}{[x^2q^2;q]_{2r}[q^3;q^2]_r}}{\displaystyle\sum_{r=0}^{\infty}\frac{[x^2q^2;q^2]_{2r}z^{2r}}{[x^2q;q]_{2r}[q;q^2]_r}+\frac{z(1-x^2q^2)}{(1-q)(1-x^2q)}\sum_{r=0}^{\infty}\frac{[x^2q^4;q^2]_{2r}z^{2r}}{[x^2q^2;q]_{2r}[q^3;q^2]_r}}
$$

$$
=\frac{1}{1}+\frac{z(1-x^2q^2)}{(1-z)-}\ \frac{x^2q(1+zq^2)}{1+}\ \frac{z(1-x^2q^4)}{(1-z)-}\ \frac{x^2q^2(1+zq^3)}{1+}\cdots. \tag{5}
$$

Again, if we take a=1 in (3.13) we get :

$$
{}_2\varphi_1\left[\begin{matrix} w,wq;q^2;z \\ w^2 \end{matrix}\right]={}_4\Phi_3\left[\begin{matrix} \sqrt{w},-\sqrt{w},\sqrt{wq},-\sqrt{wq};q;z \\ w,-w,-q \end{matrix}\right]. \tag{6}
$$

Now, making use of the result [2; (2.19)] after replacing q by q^2 we find :

$$\frac{{}_4\varphi_3\left[\begin{matrix}\sqrt{w},-\sqrt{w},\sqrt{wq},-\sqrt{wq};q;zq^2\\ w,-w,-q\end{matrix}\right]}{{}_4\varphi_3\left[\begin{matrix}\sqrt{w},-\sqrt{w},\sqrt{wq},-\sqrt{wq};q;z\\ w,-w,-q\end{matrix}\right]}$$

$$=\frac{1}{1}+\frac{(1-w)(1-wq)z}{(1-z)+}\;\frac{w^2(zq^3-1)}{1+}\;\frac{(1-wq^2)(1-wq^3)z}{(1-z)+}\;\ldots. \tag{7}$$

A number of similar other interesting results could also be deduced.

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REFERENCES

1 Agarwal, R.P. , Generalized Hypergeometric series and its applications to the theory of combinatorial analysis and partition theory (Unpublished Monograph).

2 Agarwal, R.P., Monocha, H.L. and K.Srinivasa,Rao : Selected topics in special functions; Generalized hypergeometric functions and continued fractions (R.Y.Denis & S.N. Singh pp. 173-207) Allied Publishers Limited, New Delhi, 2001.

3 Al Salam, W.A. and Verma,A., On quadratic transformations of basic series, SIAM J. Math. Anal. 15 (2) (1984), 414-421.

4 Gasper, G. and Rahman, M., Basic hypergeometric series, Cambridge University Press, Cambridge (1991).

5 Verma, A. and Jain, V.K., Some summation formulae of basic hypergeometric series, Indian J. Pure and Appl. Math. 11(8) (1980), 1021-1038.

6 Verma, A. and Jain, V.K., Certain summation formulae for q-series, J. Indian Math. Soc. 47 (1983), 71-85.