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INTEGRALS INVOLVING THE ALEPH-FUNCTION OF SEVERAL VARIABLES

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Abstract: In this paper, we evaluate four integrals involving the product of elementary special functions and the multivariable Aleph-function. The integrals are quite general in nature and from them a large number of new results can be obtained simply by specializing the parameters of the multivariable Aleph-function.

Keywords: Multivariable Aleph-function, Integral, multivariable H-function, Aleph-function of two variables, I-function of two variables.

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1. Introduction and Preliminaries

The multivariable Aleph-function is an extension of the multivariable I-function recently defined by C.K. Sharma and Ahmad [8], itself is a generalization of the multivariable H-function defined by Srivastava et al [10,11]. The multivariable Aleph-function is defined by means of the multiple contour integral :
We have,

$$\aleph(z_1,...,z_r) = \aleph^{0,n:m_1,n_1,...,m_r.n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};...;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}$$

$$\left(\begin{array}{c|c} z_1 & [(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,n}],[\tau_i(a_{ji};\alpha_{ji}^{(1)},...,\alpha_{ji}^{(r)})_{n+1,p_i}]:[(c_j^{(1)},\gamma_j^{(1)})_{1,n_1}], \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & ....[\tau_i(b_{ji};\beta_{ji}^{(1)},...,\beta_{ji}^{(r)})_{m+1,q_i}]:[(d_j^{(1)},\delta_j^{(1)})_{1,m_1}], \end{array}\right.$$

$$\left.\begin{array}{c} [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];...;;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];...;;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}...\int_{L_r}\psi(s_1,...,s_r)\prod_{k=1}^{r}\phi_k(s_k)z_k^{s_k}ds_1...ds_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, ..., s_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k)}{\sum_{i=1}^{R} \left[\tau_i \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k) \right]} \tag{1.2}$$

and

$$\phi_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} \left[\tau_{i^{(k)}} \prod_{l=m_k+1}^{q_i(k)} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{l=n_k+1}^{p_i(k)} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k) \right]} \tag{1.3}$$

where $j = 1$ to r and k=1 to r.
For more details, see Ayant [1,2,3]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as

$$|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi, \text{ where } A_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} -$$

$$\tau_{i^{(k)}} \sum_{j=n_k+1}^{p_i(k)} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{(q_i(k))} \delta_{ji^{(k)}}^{(k)} > 0, \tag{1.4}$$

with $k = 1, ..., r, i = 1, ..., R$ and $i^{(k)} = 1, ..., R^{(k)}$.
The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form,

$$\aleph(z_1, ..., z_r) = 0(|z_1|^{\alpha_1}, ..., |z_r|^{\alpha_r}), max(|z_1|, ..., |z_r|) \to 0$$

$$\aleph(z_1, ..., z_r) = 0(|z_1|^{\beta_1}, ..., |z_r|^{\beta_r}), min(|z_1|, ..., |z_r|) \to \infty$$

where $k = 1, ..., r; \alpha_k = \min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, ..., m_k$ and $\beta_k = \max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, ..., n_k$.
For convenience, we will use the following notations in this paper.

$$V = m_1, n_1; ...; m_r, n_r \tag{1.5}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, ..., p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.6}$$

$$A = \left\{(a_j; \alpha_j^{(1)}, ..., \alpha_j^{(r)})_{1,n}\right\}, \left\{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, ..., \alpha_{ji}^{(r)})_{n+1,p_i}\right\}, \left\{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\right\},$$

$$\left.\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\right\}, ..., \left\{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\right\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\Big\} \tag{1.7}$$

$$B = \left\{\tau_i(b_{ji}; \beta_{ji}^{(1)}, ..., \beta_{ji}^{(r)})_{m+1,q_i}\right\}, \left\{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\right\},$$

$$\left.\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\right\}, ..., \left\{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\right\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\Big\} \tag{1.8}$$

2. Main integrals

In this section, we have evaluated the four following integrals

First integrals

$$\int_0^\infty t^{\rho-1} J_c(at) J_d(at) J_e(2bt) \aleph_{p_i,q_i,\tau_i;R:W}^{0,n:V}\left(\begin{array}{c} x_1 t^{\rho_1} \\ \cdot \\ \cdot \\ x_n t^{\rho_n} \end{array}\right) dt = \frac{a^{c+d}}{b^{c+d+\rho} 2^{c+d}}$$

$$\sum_{r=0}^{\infty} \frac{\left(\frac{c+d+1}{2}\right)_r \left(\frac{c+d+2}{2}\right)_r \left(\frac{a^2}{b^2}\right)^r}{r!\Gamma(c+r+1)\Gamma(d+r+1)(c+d+1)_r} \aleph_{p_i+3,q_i+1,\tau_i;R:W}^{0,n+3:V}\left(\begin{array}{c|c} x_1 b^{-\rho_1} & \left[\frac{2-c-d\pm e-2r-\rho}{2} : \frac{\rho_1}{2}, ..., \frac{\rho_n}{2}\right], \left[\frac{2-c-d+e-\rho}{2} : \frac{\rho_1}{2}, ..., \frac{\rho_n}{2}\right], A \\ \cdot & \cdots \\ \cdot & \\ \cdot & \\ x_n b^{-\rho_n} & \left[\frac{2-d+e-c}{2} : \frac{\rho_1}{2}, ..., \frac{\rho_n}{2}\right], B \end{array}\right) \tag{2.1}$$

provided that $\rho_i > 0, i = 1, ..., n$ $|\arg x_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is defined by (1.4) and $Re\left[\rho + c + d + e + \sum_{i=1}^{n} \rho_i \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$ and the series occurring on the right side of (2.1) is absolutely convergent.

Second integral

$$\int_0^\infty t^{\rho-1} \cos(2\alpha t) K_\mu(t) K_\upsilon(t) \aleph_{p_i,q_i,\tau_i;R:W}^{0,n:V}\left(\begin{array}{c} x_1 t^{\rho_1} \\ \cdot \\ \cdot \\ x_n t^{\rho_n} \end{array}\right) dt = 2^{\rho-3} \sum_{r=0}^{\infty} \frac{(-)^r a^{2r}}{r!}$$

$$\aleph^{0,n+6:V}_{p_i+6,q_i+3,\tau_i;R:W}\left(\begin{array}{c|c} x_1 2^{\rho_1} & \left[\frac{2-\rho\pm\mu\pm\upsilon-2r}{2}:\frac{\rho_1}{2},...,\frac{\rho_n}{2}\right],\left[\frac{2-\rho-2r}{2}:\frac{\rho_1}{2},...,\frac{\rho_n}{2}\right],\left[\frac{1-\rho-2r}{2}:\frac{\rho_1}{2},...,\frac{\rho_n}{2}\right],A \\ . & \cdots \\ . & \\ . & (1-\rho:\rho_1,...,\rho_n),\left[\frac{2-\rho}{2}:\frac{\rho_1}{2},...,\frac{\rho_n}{2}\right],\left[\frac{1-\rho}{2}:\frac{\rho}{2},...,\frac{\rho_n}{2}\right],B \\ x_n 2^{\rho_n} & \end{array}\right) \tag{2.2}$$

provided that $\rho_i > 0, i = 1, ..., n,\ Re(\alpha) < 1\ |\arg x_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $Re\left[\rho\pm\mu\pm\upsilon+\sum_{i=1}^{n}\rho_i \min_{1\le j\le m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$ and the series occurring on the right side of (2.2) is absolutely convergent.

Third integral

$$\int_{-\infty}^{\infty} t^{\rho-1}e^{-t^2}H_{2\upsilon}(t)\aleph^{0,n:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c} x_1 t^{\rho_1} \\ . \\ . \\ x_n t^{\rho_n} \end{array}\right)dt = \frac{\sqrt{\pi}}{4^{\rho-\upsilon}}$$

$$\aleph^{0,n+1:V}_{p_i+1,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} x_1 2^{-\rho_1} & (-2\rho:2\rho_1,...,2\rho_n),A \\ . & \cdots \\ . & \\ . & (\upsilon-\rho:\rho_1,...,\rho_n),B \\ x_n 2^{-\rho_n} & \end{array}\right) \tag{2.3}$$

provided that $\rho_i > 0, i = 1, ..., n,\ |\arg x_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $Re\left[1+2\rho+2\sum_{i=1}^{n}\rho_i \min_{1\le j\le m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$.

Fourth integral

$$\int_{0}^{\infty} t^{\rho-1}e^{-\sigma t}E(\alpha_1,\beta_1::\sigma t)E(\alpha_2,\beta_2::\xi t)\aleph^{0,n:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c} x_1 t^{\rho_1} \\ . \\ . \\ x_n t^{\rho_n} \end{array}\right)dt$$

$$= \sigma^{-\rho} \sum_{r,u=0}^{\infty} \Gamma(\alpha_1 + r)\Gamma(\alpha_2 + r)\Gamma(\beta_2 + r + u) \left(\frac{\xi - \sigma}{\xi}\right)^u$$

$$\aleph^{0,n+3:V}_{p_i+3,q_i+2,\tau_i;R:W} \left(\begin{array}{c|c} x_1\sigma^{-\rho_1} & (1-\rho-\alpha_1-\alpha_2 : \rho_1, ..., \rho_n) \\ \cdot & \cdots \\ \cdot & \\ \cdot & \\ x_n\sigma^{-\rho_n} & (1-\rho-r-\alpha_1-\beta_1-\beta_2 : \rho_1, ..., \rho_n), \end{array} \right.$$

$$\left. \begin{array}{c} (1-\rho-\beta_1-\beta_2 : \rho_1, ..., \rho_n), (1-\rho-r-\alpha_1-\beta_2 : \rho_1, ..., \rho_n), A \\ \cdots \\ (1-\rho-r-u-\alpha_1-\alpha_2-\beta_2 : \rho_1, ..., \rho_n), B \end{array} \right) \tag{2.4}$$

provided that $\rho_i > 0, i = 1, ..., n,$ $Re(\xi) < \frac{1}{2},$ $Re(\sigma) > 0,$ $|\arg x_k| < \frac{1}{2}A_i^{(k)}\pi,$ where $A_i^{(k)}$ is defined by (1.4) and $Re\left[\beta_1 + \beta_2 + \rho + \sum_{i=1}^{n} \rho_i \min_{1\le j\le m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$ and $Re\left[\alpha_1 + \alpha_2 + \rho + \sum_{i=1}^{n} \rho_i \min_{1\le j\le m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$ and the series occurring on the right side of (2.4) is absolutely convergent.

Proof of (2.1)

The integral (2.1) can be established if we express the multivariable Aleph-function in the integrand on the left of (2.1) in terms of its Mellin-Barnes type contour integral (1.1), interchange the order of integrations (which is justified due to the absolute convergence of the integral involved in the process) and evaluate the inner integral with the help of the following result [6, page 350,Eq (7)].

$$\int_0^{\infty} t^{\rho-1} J_c(at)J_d(at)J_e(2bt)dt = \frac{a^{c+d}b^{-c-d-\rho}\Gamma\left(\frac{c+d+e+\rho}{2}\right)}{2^{c+d}\Gamma\left(1-\frac{1}{2}c-\frac{1}{2}d-\frac{1}{2}e-\frac{1}{2}\rho\right)} \times$$

$${}_4F_3 \left[\begin{array}{c|c} \frac{c+d+1}{2}, \frac{c+d+1}{2}, \frac{c+d+e+\rho}{2}, \frac{c+d-e+\rho}{2} & \frac{a^2}{b^2} \\ c+1, d+1, c+d+1 & \end{array} \right] \tag{2.5}$$

Now, writing series expansion for ${}_4F_3$ changing the order of integration ans summation and interpreting the result thus obtained with the help of (1.1), we obtain the right hand side of (2.1). The integrals (2.2) to (2.4) can be established in the similar manner with the difference that we use the results [6, page 371, Eq (51)], [5, page 93, (2.2.7)] and [5, page 90, (2.2.2)] respectively instead of (2.5).

Remarks. We obtain the same relations with the multivariable H-function defined by Srivastava and Panda [10,11], see Bohara and Jain [4] for more details.

If $r = 2$, the multivariable Aleph-function reduces to Aleph-function of two variables defined by Sharma [7], and we obtain the same relations.

If $r = 2$ and $\tau_i, \tau_{i'}, \tau_{i''} \to 1$, the multivariable Aleph-function reduces to I-function of two variables defined by Sharma and Mishra [9] and we have the similar formulae.

4. Conclusion

Specializing the parameters of the multivariable Aleph-function, we can obtain a large number of news and knowns integrals involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

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