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A NOTE ON THETA HYPERGEOMETRIC SERIES

Satya Prakash Singh[1], Bindu Prakash Mishra[2], Mohmmad Shahjade[3] and Vijay Yadav[4]

[1]Department of Mathematics,
T.D.P.G. College, Jaunpur-222002 (U.P.), INDIA
E-mail: snsp39@gmail.com

[2]Department of Mathematics,
M.D. College, Parel, Mumbai, INDIA
E-mail: bindu1962@gmail.com

[3] Department of Mathematics,
MANUU Poly. 8th Cross, 1st Stage,
3rd Block, Nagarbhavi, Banglore-72, INDIA
E-mail: mohammadshahjade@gmail.com

[4]Department of Mathematics,
S.P.D.T. College, Andheri(E), Mumbai-400059, INDIA
E-mail: vijaychottu@yahoo.com

Abstract: In this article, making use of Bailey transform and certain known identities, we have established certain transformation formulas for elliptic hypergeometric series

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1. Introduction, Notations and Definitions

In a path-breaking paper, Frankel and Turaev [1] introduced elliptic analogues of very well-poised basic hypergeometric series. Elliptic hypergeometric series and their extensions to theta hypergeometric series has become an increasingly active area of research now these days. So for, many formulae for very well-poised basic hypergeometric series have already been extended to the elliptic setting. Some formulae for multi-basic elliptic hypergeometric series appeared in the work of Warnaar [7]. In this paper, using certain identities we have established transformation

formulae for the theta hypergeometric series.
A modified Jacobi's theta function with argument x and nome p is defined by,

$$\theta(x;p) = [x;p]_\infty [p/x;p]_\infty = [x, p/x; p]_\infty$$

Also,

$$\theta(x_1, x_2, ..., x_r; p) = \theta(x_1;p)\theta(x_2;p)...\theta(x_r;p),$$

and

$$[x;p]_\infty = \prod_{r=0}^{\infty}(1 - xp^r). \tag{1.1}$$

Following Gasper and Rahman [2; Chapter 11] theta shifted factorial is defined by,

$$[a;q,p]_n = \theta(a;p)\theta(aq;p)...\theta(aq^{n-1};p),$$

with

$$\theta[a;q,p]_0 = 1$$

and

$$[a;q,p]_{-n} = \frac{(-)^n q^{n(n+1)/2}}{a^n [q/a;q,p]_n}.$$

Also,

$$[a_1, a_2, ..., a_r; q, p]_n = [a_1;q,p]_n [a_2;q,p]_n ... [a_r;q,p]_n, \tag{1.2}$$

where $a_1, a_2, ..., a_r \neq 0$.
Following Spiridonov [5], we define an ${}_{r+1}E_r$, a theta hypergeometric series with base q and nome p by,

$${}_{r+1}E_r \left[\begin{array}{l} a_1, a_2, ..., a_{r+1}; q, p; z \\ b_1, b_2, ..., b_r \end{array} \right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_{r+1}; q, p]_n z^n}{[q, b_1, b_2, ..., b_r; q, p]_n} \tag{1.3}$$

where $a's$ and $b's$ are never zero. If z and $a's$ and $b's$ are independent of p then

$$Lim_{p\to 0} \; {}_{r+1}E_r \left[\begin{array}{l} a_1, a_2, ..., a_{r+1}; q, p; z \\ b_1, b_2, ..., b_r \end{array} \right]$$

$$= {}_{r+1}E_r \left[\begin{array}{l} a_1, a_2, ..., a_{r+1}; q, 0; z \\ b_1, b_2, ..., b_r \end{array} \right]$$

$$= {}_{r+1}\Phi_r \left[\begin{array}{l} a_1, a_2, ..., a_{r+1}; q; z \\ b_1, b_2, ..., b_r \end{array} \right]. \tag{1.4}$$

For $0 < |q|, |p| < 1$ and any $Re[0, \infty]$ there is a non terminating ${}_{r+1}E_r$ series with radius of convergence R. In general we call a series (unilateral or bilateral) $\sum c_n$ an elliptic hypergeometric series if $g(n) = \frac{c_{n+1}}{c_n}$ is an elliptic function of n with n considered as a complex variable i.e. $g(x)$ is doubly periodic meromorphic function of the complex variable x. From (1.3) we have,

$$c_n = \frac{[a_1, a_2, ..., a_{r+1}; q, p]_n z^n}{[q, b_1, b_2, ..., b_r; q, p]_n}; \qquad c_{n+1} = \frac{[a_1, a_2, ..., a_{r+1}; q, p]_{n+1} z^{n+1}}{[q, b_1, b_2, ..., b_r; q, p]_{n+1}}$$

and hence

$$g(n) = \frac{c_{n+1}}{c_n} = \frac{\theta(a_1 q^n, a_2 q^n, ..., a_{r+1} q^n; p) z}{\theta(q^{n+1}, b_1 q^n, b_2 q^n, ..., b_r q^n; p)}$$

we have

$$g(x) = \frac{\theta(a_1 q^x, a_2 q^x, ..., a_{r+1} q^x; p) z}{\theta(q^{x+1}, b_1 q^x, b_2 q^x, ..., b_r q^x; p)} \tag{1.5}$$

is a meromorphic function of x. If $a_1.a_2....a_{r+1} = (b_1.b_2...b_r)q$, then $g(x)$ is an elliptic function of x, a doubly periodic meromorphic function.

The elliptic hypergeometric function ${}_{r+1}E_r$ is called a well-poised if $qa_1 = a_2 b_1 = a_3 b_2 = ... = a_{r+1} b_r$ in which case the balancing condition $b_1 b_2 ... b_r q = a_1 a_2 ... a_{r+1}$ reduces to

$$(a_1 q)^{r+1} = (a_1 a_2 ... a_{r+1})^2 \tag{1.6}$$

Also,

$$\frac{\theta(aq^{2n}; p)}{\theta(a; p)} = \frac{[q\sqrt{a}, -q\sqrt{a}, q\sqrt{a/p}, -q\sqrt{ap}; q, p]_n}{[\sqrt{a}, -\sqrt{a}, \sqrt{ap}, -\sqrt{a/p}; q, p]_n} (-q)^{-n}$$

which is the elliptic analogue of the quotient

$$\frac{1 - aq^{2n}}{1 - a} = \frac{[q\sqrt{a}, -q\sqrt{a}; q]_n}{[\sqrt{a}, -\sqrt{a}; q]_n},$$

This is the very well-poised part of ${}_{r+1}W_r$ series.
Following Spiridonov [5], the very well-poised theta hypergeometric series is defined by

$${}_{r+1}V_r[a_1; a_6, a_7, ..., a_{r+1}; q, p; z] = \sum_{n=0}^{\infty} \frac{\theta(aq^{2n}; p)[a_1, a_6, a_7, ..., a_{r+1}; q, p]_n (zq)^n}{\theta(a; p)[a_1 q/a_6, ..., a_1 q/a_{r+1}; q, p]_n}$$

$$= {}_{r+1}E_r \left[\begin{matrix} a_1, q\sqrt{a_1}, -q\sqrt{a_1}, q\sqrt{a_1/p}, -q\sqrt{a_1 p}, a_6, ..., a_{r+1}; q, p \\ q, \sqrt{a_1}, -\sqrt{a_1}, \sqrt{a_1 p}, -\sqrt{a_1/p}, a_1 q/a_6, ..., a_1 q/a_{r+1} \end{matrix} \right] \tag{1.7}$$

If the argument z in ${}_{r+1}V_r$ is 1, then we suppress 1 and denote it by

$$ {}_{r+1}V_r[a_1, a_6, a_7, ..., a_{r+1}; q, p] \tag{1.8}$$

Following Gasper and Schlosser [3], we define bi-basic and bi-nome theta hypergeometric series by,

$$ {}_{r+s+1}E_{r+s}\left[\begin{array}{l} a_1, a_2, ..., a_{r+1}; A_1, A_2, ..., A_s; (q,p); (Q,P); z \\ b_1, b_2, ..., b_s; B_1, B_2, ..., B_s \end{array}\right]$$

$$= \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_{r+1}; q, p]_n [A_1, A_2, ..., A_s; q, P]_n z^n}{[q, b_1, b_2, ..., b_r; q; p]_n [B_1, B_2, ..., B_s; Q, P]_n}, \tag{1.9}$$

where max $(|z|, |p|, |q|, |P|, |Q|) < 1$.
We shall make use of following summation formulae in our analysis,

$$ {}_{10}V_9\left[a, b, c, d, e, q^{-n}; q, p\right] = \frac{[aq, aq/bc, aq/bd, aq/cd; q, p]_n}{[aq/b, aq/c, aq/d, aq/bcd; q, p]_n} \tag{1.10}$$

[2; (11.2.25), p.307]

provided $bcde = a^2q^{n+1}$.
If we take $e = aq^{n+1}$ in the above, we get

$$ {}_8V_7[a; b, c, a/bc; q, p]_n = \frac{[aq, aq/bc, bq, cq; q, p]_n}{[q, aq/b, aq/c, bcd; q, p]_n} \tag{1.11}$$

where ${}_8V_7\left[a; b, c, a/bc; q, p\right]_n$ is a truncated theta hypergeometric function defined by

$$ {}_8V_7[a; b, c, d; q, p; z]_r = \sum_{n=0}^{r} \frac{\theta(aq^{2n}; p)[a, b, c, d; q, p]_n (zq)^n}{\theta(a; p)[q, aq/b, aq/c, aq/d; q, p]_n}. \tag{1.12}$$

Also we have

$$ {}_{10}V_9[ap; b, c, apq/d, ep, q^{-n}; q; p^2] = \frac{[apq, apq/bc, d/b, d/c; q, p^2]_n}{[apq/b, apq/c, d, d/bc; q, p^2]_n} \tag{1.13}$$

[2; (11.4.11), p.323]

provided $bce = adq^n$, $|p| < 1$.
Setting $e = aq^{n+1}$ in (1.13) we have

$$ {}_8V_7[ap; b, c, ap/bc; q; p^2]_n = \frac{[apq, apq/bc, bq, cq; q, p^2]_n}{[apq/b, apq/c, q, bcq; q, p^2]_n} \tag{1.14}$$

Again, we have

$$\sum_{k=0}^{n}\frac{\theta\{ad(rst/q)^k,(b/d)(r/q)^k,(c/d)(s/q)^k,(ad/bc)(t/q)^k;p\}}{\theta(ad,b/d,c/d,ad/bc;p)}\times$$

$$\times\frac{[a;rst/q^2,p]_k[b;r,p]_k[c;s,p]_k[ad^2/bc;t,p]_kq^k}{[dq;q,p]_k[adst/bq;st/q,p]_k[adrt/cq;rt/q,p]_k[bcrs/dq;rs/q,p]_k}$$

$$=\frac{\theta(a,b,c,ad^2/bc;p)[arst/q^2;rst/q^2,p]_n[br;r,p]_n[cs;s,p]_n}{d\theta(ad,b/d,c/d,ad/bc;p)[dq;q,p]_n[adst/bq;st/q,p]_n[adrt/cq;rt/q,p]_n}\times$$

$$\times\frac{[ad^2t/bc;t,p]_n}{[bcrs/dq;rs/q,p]_n}-\frac{\theta(d,ad/b,ad/c,bc/d;p)}{d\theta(ad,b/d,c/d,ad/bc;p)} \tag{1.15}$$

[2; (11.6.9), p.327]

As $d\to 1$, the above transformation leads to,

$$\sum_{k=0}^{n}\frac{\theta\{a(rst/q)^k,b(r/q)^k,c(s/q)^k,(a/bc)(t/q)^k;p\}}{\theta(a,b,c,a/bc;p)}\times$$

$$\times\frac{[a;rst/q^2,p]_k[b;r,p]_k[c;s,p]_k[a/bc;t,p]_kq^k}{[q;q,p]_k[ast/bq;st/q,p]_k[art/cq;rt/q,p]_k[bcrs/q;rs/q,p]_k}$$

$$=\frac{[arst/q^2;rst/q^2,p]_n[br;r,p]_n[cs;s,p]_n[at/bc;t,p]_n}{[q;q,p]_n[ast/bq;st/q,p]_n[art/cq;rt/q,p]_n[bcrs/q;rs/q,p]_n} \tag{1.16}$$

In order to establish the transformation formulae we shall make use of following identities.
In 1944, Bailey stated following theorem which is known as Bailey's transform,
If

$$\beta_n=\sum_{r=0}^{n}\alpha_r u_{n-r}v_{n+r} \tag{1.17}$$

and

$$\gamma_n=\sum_{r=n}^{\infty}\delta_r u_{r-n}v_{n+r}=\sum_{r=0}^{\infty}\delta_{r+n}u_r v_{r+2n} \tag{1.18}$$

then under suitable condition of convergence,

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n=\sum_{n=0}^{\infty}\beta_n\delta_n \tag{1.19}$$

[4; (2.4.1;2.4.2 and 2.4.3), p.58,59]

where α_r, δ_r, u_r and v_r are any rational functions of r alone.
If we choose $u_r = v_r = 1$ and $\delta_r = z^r$ in (1.17) and (1.18) then Bailey's transform takes the form,
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r \tag{1.20}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} z^{r+n} \tag{1.21}$$

then,

$$\sum_{n=0}^{\infty} \alpha_n z^n = (1-z) \sum_{n=0}^{\infty} \beta_n z^n \tag{1.22}$$

provided the series on both sides are convergent.
We shall also make use of the following known series transformations,

$$\sum_{k=0}^{n} \lambda_k \sum_{j=0}^{n-k} A_j = \sum_{k=0}^{n} A_k \sum_{j=0}^{n-k} \lambda_j \tag{1.23}$$

[2; (11.6.18), p.329]

(λ_r and A_r being two arbitrary sequences)
and

$$\sum_{m=0}^{n} \delta_m \sum_{r=o}^{m} \alpha_r = \sum_{k=0}^{n} \alpha_k \sum_{m=0}^{n} \delta_m - \sum_{r=0}^{n-1} \alpha_{r+1} \sum_{m=0}^{r} \delta_m \tag{1.24}$$

[6; (2), p.1539]

(α_r and δ_r being two arbitrary sequences)

2. Main Results

In this section we shall establish transformation formulae for theta hypergeometric series.

(i) If we choose,

$$\alpha_r = \frac{\theta(aq^{2r};p)[a,b,c,a/bc;q,p]_r q^r}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_r}$$

in (1.20) and use (1.11) we get

$$\beta_n = \frac{[aq,aq/bc,bq,cq;q,p]_n}{[q,aq/b,aq/c,bcq;q,p]_n}$$

substituting this values of α_n and β_n in (1.22) we get

$$ {}_8V_7[a;b,c,a/bc;q,p;z] = (1-z)\ {}_4E_3\left[\begin{array}{l} aq, aq/bc, bq, cq; q, p; z \\ aq/b, aq/c, bcq \end{array}\right], \tag{2.1} $$

where either b or c is of the form q^{-n}

(ii) Next, setting

$$ \alpha_r = \frac{\theta(apq^{2r};p^2)[ab,b,c,ap/bc;q,p^2]_r q^r}{\theta(ap;p^2)[q,apq/b,apq/c,bcq;q,p^2]_r} $$

in (1.20) and applying (1.14), we get

$$ \beta_n = \frac{[apq,apq/bc,bq,cq;q,p^2]_n}{[q,apq/b,apq/c,bcq;q,p^2]_n} $$

with the above values of α_n and β_n in (1.22), we get

$$ {}_8V_7[ap;b,c,ap/bc;q,p^2;z] = (1-z)\ {}_4E_3\left[\begin{array}{l} apq, apq/bc, bq, cq; q, p^2; z \\ apq/b, apq/c, bcq \end{array}\right], \tag{2.2} $$

where either b or c is of the form q^{-n}.

Replacing a by a/p in the above, we get

$$ {}_8V_7[a;b,c,a/bc;q,p^2;z] = (1-z)\ {}_4E_3\left[\begin{array}{l} aq, aq/bc, bq, cq; q, p^2; z \\ aq/b, aq/c, bcq \end{array}\right]. \tag{2.3} $$

Again, (2.2) with a replaced by a/q yields

$$ {}_8V_7[ap/q;b,c,ap/bcq;q,p^2;z] = (1-z)\ {}_4E_3\left[\begin{array}{l} ap, ap/bc, bq, cq; q, p^2; z \\ ap/b, ap/c, bcq \end{array}\right]. \tag{2.4} $$

(iii) Further, if we set

$$ \alpha_k = \frac{\theta\{ad(rst/q)^k,(b/d)(r/q)^k,(c/d)(s/q)^k,(ad/bc)(t/q)^k;p\}}{\theta(ad,b/d,c/d,ad/bc;p)}\times $$

$$ \times\frac{[a;rst/q^2,p]_k[b;r,p]_k[c;s,p]_k[ad^2/bc;t,p]_k q^k}{[dq;q,p]_k[adst/bq;st/q,p]_k[adrt/cq;rt/q,p]_k[bcrs/dq;rs/q,p]_k} $$

in (1.20) and use (1.15), we get

$$ \beta_n = \frac{\theta(a,b,c,ad^2/bc;p)[arst/q^2;rst/q^2,p]_n[br;r,p]_n[cs;s,p]_n}{d\theta(ad,b/d,c/d,ad/bc;p)[dq;q,p]_n[adst/bq;st/q,p]_n[adrt/cq;rt/q,p]_n}\times $$

$$\times \frac{[ad^2t/bc;t,p]_n}{[bcrs/dq;rs/q,p]_n} - \frac{\theta(d,ad/b,ad/c,bc/d;p)}{d\theta(ad,b/d,c/d,ad/bc;p)}$$

Substituting the above values in (1.22), we get

$$\sum_{n=0}^{\infty} \frac{\theta\{ad(rst/q)^n,(b/d)(r/q)^n,(c/d)(s/q)^n,(ad/bc)(t/q)^n;p\}}{\theta(ad,b/d,c/d,ad/bc;p)} \times$$

$$\times \frac{[a;rst/q^2,p]_n[b;r,p]_n[c;s,p]_n[ad^2/bc;t,p]_n qz^n}{[dq;q,p]_n[adst/bq;st/q,p]_n[adrt/cq;rt/q,p]_n[bcrs/dq;rs/q,p]_n}$$

$$= (1-z)\frac{\theta(a,b,c,ad^2/bc;p)}{d\theta(ad,b/d,c/d,ad/bc;p)} \sum_{n=0}^{\infty} \frac{[arst/q^2;rst/q^2,p]_n[br;r,p]_n[cs;s,p]_n}{[dq;q,p]_n[adst/bq;st/q,p]_n[adrt/cq;rt/q,p]_n}$$

$$\times \frac{[ad^2t/bc;t,p]_n}{[bcrs/dq;rs/q,p]_n} - \frac{\theta(d,ad/b,ad/c,bc/d;p)}{d\theta(ad,b/d,c/d,ad/bc;p)}, \tag{2.5}$$

where either b or c is of the form q^{-n}

Now, setting $d = 1$ in (2.5), we get

$$\sum_{n=0}^{\infty} \frac{\theta\{a(rst/q)^n,b(r/q)^n,c(s/q)^n,(a/bc)(t/q)^n;p\}}{\theta(a,b,c,a/bc;p)} \times$$

$$\times \frac{[a;rst/q^2,p]_n[b;r,p]_n[c;s,p]_n[a/bc;t,p]_n qz^n}{[q;q,p]_n[ast/bq;st/q,p]_n[art/cq;rt/q,p]_n[bcrs/q;rs/q,p]_n}$$

$$= (1-z)\sum_{n=0}^{\infty} \frac{[arst/q^2;rst/q^2,p]_n[br;r,p]_n[cs;s,p]_n}{[q;q,p]_n[ast/bq;st/q,p]_n[art/cq;rt/q,p]_n} \frac{[at/bc;t,p]_n}{[bcrs/q;rs/q,p]_n} \tag{2.6}$$

where either b or c is of the form q^{-n}

With $r = s = t = q$ in (2.6), we get

$$\sum_{n=0}^{\infty} \frac{\theta(aq^{2n};p)[a;q,p]_n[b;q,p]_n[c;q,p]_n[a/bc;q,p]_n(zq)^n}{\theta(a;p)[q;q,p]_n[aq/b;q,p]_n[aq/c;q,p]_n[bcq;q,p]_n}$$

$$= (1-z)\sum_{n=0}^{\infty} \frac{[aq,bq,cq,aq/bc;q,p]_n z^n}{[q,aq/b,aq/c,bcq;q,p]_n} \tag{2.7}$$

where either b or c is of the form q^{-n}

which is precisely (2.1).

Taking $b = c = a$ in (2.7) we have,

$$\sum_{n=0}^{\infty} \frac{\theta(aq^{2n};p)[a;q,p]_n^3[1/a;q,p]_n(zq)^n}{\theta(a;p)[q;q,p]_n^3[a^2q;q,p]_n}$$

$$= (1-z)\sum_{n=0}^{\infty}\frac{[aq;q,p]_n^3 z^n [q/a;q,p]_n z^n}{[q;q,p]_n^3[a^2q;q;p]_n} \tag{2.8}$$

provided a is of the form q^m.

(iv) Choosing

$$A_r = \frac{\theta(\alpha Q^{2r};P)[\alpha,\beta,\gamma,\alpha/\beta\gamma;Q,P]_r Q^r}{\theta(\alpha;P)[Q,\alpha Q/\beta,\alpha Q/\gamma,\beta\gamma Q;Q,P]_r}$$

and

$$\lambda_r = \frac{\theta(aq^{2r};p)[a,b,c,a/bc;q,p]_r q^r}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_r}$$

in (1.23) and using (1.11) we get,

$$\frac{\left[\alpha Q,\frac{\alpha Q}{\beta\gamma},\beta Q,\gamma Q;Q,P\right]_n}{\left[Q,\frac{\alpha Q}{\beta},\frac{\alpha Q}{\gamma},\beta\gamma Q,Q,P\right]_n}\,{}_{12}E_{11}\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},q\sqrt{a/p},-q\sqrt{ap},b,c,a/bc; \\ \sqrt{a},-\sqrt{a},\sqrt{ap},-\sqrt{a/p},aq/b,aq/c,bcq; \end{matrix}\right.$$

$$\left.\begin{matrix} ;\frac{\beta}{\alpha}Q^{-n},\frac{\gamma}{\alpha}Q^{-n},\frac{Q^{-n}}{\beta\gamma},Q^{-n};(q,p),(Q,P);-1 \\ ;\frac{Q^{-n}}{\alpha},\frac{\beta\gamma Q^{-n}}{\alpha},\frac{Q^{-n}}{\beta},\frac{Q^{-n}}{\gamma} \end{matrix}\right]$$

$$= \frac{[aq,bq,cq,aq/bc;q,p]_n}{[q,aq/b,aq/c,bcq;q,p]_n}\,{}_{12}E_{11}\left[\begin{matrix} \alpha,Q\sqrt{\alpha},-Q\sqrt{\alpha},Q\sqrt{\frac{\alpha}{P}},-Q\sqrt{\alpha P},\beta,\gamma,\frac{\alpha}{\beta\gamma} \\ \sqrt{\alpha},-\sqrt{\alpha},\sqrt{\alpha P},-\sqrt{\frac{\alpha}{P}},\frac{\alpha Q}{\beta},\frac{\alpha Q}{\gamma},\beta\gamma Q \end{matrix}\right.$$

$$\left.\begin{matrix} ;\frac{b}{a}q^{-n},\frac{c}{a}q^{-n},\frac{q^{-n}}{bc},q^{-n};(Q,P),(q,p);-1 \\ ;\frac{q^{-n}}{a},\frac{bcq^{-n}}{a},\frac{q^{-n}}{b},\frac{q^{-n}}{c} \end{matrix}\right] \tag{2.9}$$

(v) Next setting

$$\lambda_r = \frac{\theta(aq^{2r};p)[a,b,c,a/bc;q,p]_r q^r}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_r}$$

and

$$A_r = \frac{\theta(\alpha p_1 q_1^{2r};p_1^2)[\alpha p_1,\beta,\gamma,\alpha p_1/\beta\gamma;q,p_1^2]_r q_1^r}{\theta(\alpha p_1;p_1^2)[q_1,\alpha p_1 q_1/\beta,\alpha p_1 q_1/\gamma,\beta\gamma q_1;q_1,p_1^2]_r}$$

in (1.23) and using (1.11) and (1.14), we get

$$\frac{[\alpha p_1 q_1,\alpha p_1 q_1/\beta\gamma,\beta q_1,\gamma q_1;q_1,p_1^2]_n}{[\alpha p_1 q_1/\beta,\alpha q_1 p_1/\gamma,\beta\gamma q_1,q_1;q_1,p_1^2]_n}\times$$

$$\times\ {}_{12}E_{11}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q\sqrt{\frac{a}{p}}, -q\sqrt{ap}, b, c, \frac{a}{bc}; \\ \sqrt{a}, -\sqrt{a}, \sqrt{ap}, -\sqrt{\frac{a}{p}}, \frac{aq}{b}, \frac{aq}{c}, bcq; \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{\beta q_1^{-n}}{\alpha p_1}, \frac{\gamma q_1^{-n}}{\alpha p_1}, \frac{q_1^{-n}}{\beta\gamma}, q_1^{-n}; (q,p), (q_1,p_1); -1 \\ ; \frac{q_1^{-n}}{\alpha}, \frac{\beta\gamma q_1^{-n}}{\alpha}, \frac{q_1^{-n}}{\beta}, \frac{q_1^{-n}}{\gamma} \end{matrix}\right]$$

$$= \frac{[aq, bq, cq, aq/bc; q, p]_n}{[q, aq/b, aq/c, bcq; q, p]_n}\times$$

$$\times\ {}_{12}E_{11}\left[\begin{matrix} \alpha p_1, q_1\sqrt{\alpha p_1}, -q_1\sqrt{\alpha p_1}, q_1\sqrt{\alpha/p_1}, q_1\sqrt{\alpha p_1^3}, \beta, \gamma, \alpha p_1/\beta\gamma; \\ \sqrt{\alpha p_1}, -\sqrt{\alpha p_1}, \sqrt{\alpha p_1^3}, -\sqrt{\alpha/p_1}, \alpha p_1 q_1/\beta, \alpha p_1 q_1/\gamma, \beta\gamma q_1; \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{bq^{-n}}{a}, \frac{cq^{-n}}{a}, \frac{q^{-n}}{bc}, q^{-n}; (q_1,p_1), (q,p); -1 \\ ; \frac{q^{-n}}{a}, \frac{bcq^{-n}}{a}, \frac{q^{-n}}{b}, \frac{q^{-n}}{c} \end{matrix}\right] \tag{2.10}$$

(vi) Choosing

$$\lambda_k = \frac{\theta(aq^{2k};p)[a,b,c,a/bc;q,p]_k q^k}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_k}$$

and

$$A_k = \frac{\theta\{\alpha\delta(rst/q_1)^k, \frac{\beta}{\delta}(r/q_1)^k, \frac{\gamma}{\delta}(s/q_1)^k, (\alpha\delta/\beta\gamma)(t/q_1)^k; p\}}{\theta(\alpha\delta, \beta/\delta, \gamma/\delta, \alpha\delta/\beta\gamma; p_1)}$$

$$\times\frac{[\alpha; rst/q_1^2, p_1]_k[\beta; r, p_1]_k[\gamma; s, p_1]_k[\alpha\delta^2/\beta\gamma; t, p_1]_k q_1^k}{[\delta q_1; q_1, p_1]_k[\alpha\delta st/\beta q_1; st/q_1, p_1]_k[\alpha\delta rt/\gamma q_1; rt/q_1, p_1]_k[\beta\gamma rs/q_1; sr/q_1, p_1]_k}$$

in (1.23) and using (1.11) and (1.15), we get

$$\frac{\theta\left(\alpha, \beta, \gamma, \dfrac{\alpha\delta^2}{\beta\gamma}; p_1\right)}{\delta\theta\left(\alpha\delta, \dfrac{\beta}{\delta}, \dfrac{\gamma}{\delta}, \dfrac{\alpha\delta}{\beta\gamma}; p_1\right)}\frac{\left(\alpha\dfrac{rst}{q_1^2}; \dfrac{rst}{q_1^2}, p_1\right)_n(\beta r; r, p_1)_n(\gamma s; s, p_1)_n\left(\dfrac{\alpha\delta^2 t}{\beta\gamma}; t, p_1\right)_n}{(\delta q_1; q_1, p_1)_n\left(\dfrac{\alpha\delta}{\beta}\dfrac{st}{q_1}; \dfrac{st}{q_1}, p_1\right)_n\left(\dfrac{\alpha\delta}{\gamma}\dfrac{rt}{q_1}; \dfrac{rt}{q_1}, p_1\right)_n\left(\dfrac{\beta\gamma}{\delta}\dfrac{rs}{q_1}; \dfrac{rs}{q_1}, p_1\right)_n}$$

$$\times\sum_{k=0}^{n}\frac{\theta(aq^{2k};p)\left[a,b,c,\dfrac{a}{bc};q,p\right]_k q^k\left[\dfrac{q_1^{-n}}{\delta}; q_1, p_1\right]_k\left[\dfrac{\beta}{\alpha\delta}\left(\dfrac{st}{q_1}\right)^{-n}; \dfrac{st}{q_1}, p_1\right]_k}{\theta(a,p)\left[q,\dfrac{aq}{b},\dfrac{aq}{c},bcq;q,p\right]_k\left[\dfrac{1}{\alpha}\left(\dfrac{rst}{q_1^2}\right)^{-n}; \dfrac{rst}{q_1^2}, p_1\right]_k\left[\dfrac{1}{\beta}r^{-n}; r, p_1\right]_k}\times$$

$$\frac{\left[\frac{\gamma}{\alpha\delta}\left(\frac{rt}{q_1}\right)^{-n};\frac{rt}{q_1},p_1\right]_k\left[\frac{\delta}{\beta\gamma}\left(\frac{rs}{q_1}\right)^{-n};\frac{rs}{q_1},p_1\right]_k}{\left[\frac{1}{\gamma}s^{-n};s,p_1\right]_k\left[\frac{\beta\gamma}{\alpha\delta^2}t^{-n};t,p_1\right]_k}-\frac{\left[\delta,\frac{\alpha\delta}{\beta},\frac{\alpha\delta}{\gamma},\frac{\beta\gamma}{\delta};p_1\right]}{\delta\theta\left[\alpha\delta,\frac{\beta}{\delta},\frac{\gamma}{\delta},\frac{\alpha\delta}{\beta\gamma};p_1\right]}\times$$

$$\times\frac{\left[aq,\frac{aq}{bc},bq,cq;q,p\right]_n}{\left[q,\frac{aq}{b},\frac{aq}{c},bcq;q,p\right]_n}$$

$$=\frac{\left[aq,\frac{aq}{bc},bq,cq;q,p\right]_n}{\left[q,\frac{aq}{b},\frac{aq}{c},bcq;q,p\right]_n}\sum_{k=0}^{n}\frac{\theta\left\{\alpha\delta\left(\frac{rst}{q_1}\right)^k,\frac{\beta}{\delta}\left(\frac{r}{q_1}\right)^k,\frac{\gamma}{\delta}\left(\frac{s}{q_1}\right)^k,\frac{\alpha\delta}{\beta\gamma}\left(\frac{t}{q_1}\right)^k;p_1\right\}}{\theta\left(\alpha\delta,\frac{\beta}{\delta},\frac{\gamma}{\delta},\frac{\alpha\delta}{\beta\gamma};p_1\right)}$$

$$\times\frac{\left(\alpha\frac{rst}{q_1^2},p_1\right)_k(\beta;r,p_1)_k(\gamma;s,p_1)_k\left(\frac{\alpha\delta^2}{\beta\gamma};t,p_1\right)_k}{(\delta q_1;q_1,p_1)_k\left(\frac{\alpha\delta}{\beta}\frac{st}{q_1};\frac{st}{q_1},p_1\right)_k\left(\frac{\alpha\delta}{\gamma}\frac{rt}{q_1};\frac{rt}{q_1},p_1\right)_k\left(\frac{\beta\gamma}{\delta}\frac{rs}{q_1};\frac{rs}{q_1},p_1\right)_k}$$

$$\times\frac{\left[\frac{b}{a}q^{-n},\frac{c}{a}q^{-n},\frac{q^{-n}}{bc},q^{-n};q,p\right]_k}{\left[\frac{q^{-n}}{a},\frac{bcq^{-n}}{a},\frac{q^{-n}}{b},\frac{q^{-n}}{c};q,p\right]_k}. \tag{2.11}$$

For $\delta=1$, in (2.11) yields

$$\frac{\left(\alpha\frac{rst}{q_1^2};\frac{rst}{q_1^2},p_1\right)_n(\beta r;r,p_1)_n(\gamma s;s,p_1)_n\left(\frac{\alpha t}{\beta\gamma};t,p_1\right)_n}{(q_1;q_1,p_1)_n\left(\frac{\alpha}{\beta}\frac{st}{q_1};\frac{st}{q_1},p_1\right)_n\left(\frac{\alpha}{\gamma}\frac{rt}{q_1};\frac{rt}{q_1},p_1\right)_n\left(\beta\gamma\frac{rs}{q_1};\frac{rs}{q_1},p_1\right)_n}$$

$$\times\sum_{k=0}^{n}\frac{\theta(aq^{2k};p)\left[a,b,c,\frac{a}{bc};q,p\right]_k q^k\left[q_1^{-n};q_1,p_1\right]_k\left[\frac{\beta}{\alpha}\left(\frac{st}{q_1}\right)^{-n};\frac{st}{q_1},p_1\right]_k}{\theta(a,p)\left[q,\frac{aq}{b},\frac{aq}{c},bcq;q,p\right]_k\left[\frac{1}{\alpha}\left(\frac{rst}{q_1^2}\right)^{-n};\frac{rst}{q_1^2},p_1\right]_k\left[\frac{1}{\beta}r^{-n};r,p_1\right]_k}\times$$

$$\frac{\left[\frac{\gamma}{\alpha}\left(\frac{rt}{q_1}\right)^{-n};\frac{rt}{q_1},p_1\right]_k\left[\frac{1}{\beta\gamma}\left(\frac{rs}{q_1}\right)^{-n};\frac{rs}{q_1},p_1\right]_k}{\left[\frac{1}{\gamma}s^{-n};s,p_1\right]_k\left[\frac{\beta\gamma}{\alpha}t^{-n};t,p_1\right]_k}$$

$$=\frac{\left[aq,\frac{aq}{bc},bq,cq;q,p\right]_n}{\left[q,\frac{aq}{b},\frac{aq}{c},bcq;q,p\right]_n}\sum_{k=0}^{n}\frac{\theta\left\{\alpha\left(\frac{rst}{q_1}\right)^k,\beta\left(\frac{r}{q_1}\right)^k,\gamma\left(\frac{s}{q_1}\right)^k,\frac{\alpha}{\beta\gamma}\left(\frac{t}{q_1}\right)^k;p_1\right\}}{\theta\left(\alpha,\beta,\gamma,\frac{\alpha}{\beta\gamma};p_1\right)}$$

$$\times\frac{\left(\alpha;\frac{rst}{q_1^2},p_1\right)_k(\beta;r,p_1)_k(\gamma;s,p_1)_k\left(\frac{\alpha}{\beta\gamma};t,p_1\right)_k q_1^k}{(q_1;q_1,p_1)_k\left(\frac{\alpha}{\beta}\frac{st}{q_1};\frac{st}{q_1},p_1\right)_k\left(\frac{\alpha}{\gamma}\frac{rt}{q_1};\frac{rt}{q_1},p_1\right)_k\left(\beta\gamma\frac{rs}{q_1};\frac{rs}{q_1},p_1\right)_k}$$

$$\times\frac{\left[\frac{b}{a}q^{-n},\frac{c}{a}q^{-n},\frac{q^{-n}}{bc},q^{-n};q,p\right]_k}{\left[\frac{q^{-n}}{a},\frac{bcq^{-n}}{a},\frac{q^{-n}}{b},\frac{q^{-n}}{c};q,p\right]_k}. \tag{2.12}$$

Taking $r=s=q=t=q_1$ and $p_1=p$ in (2.12) we get,

$$\frac{[\alpha q;q,p]_n[\beta q;q,p]_n[\gamma q;q,p]_n\left[\frac{\alpha q}{\beta\gamma};q,p\right]_n}{\left[\frac{\alpha q}{\beta},\frac{\alpha q}{\gamma},\beta\gamma q;q,p\right]_n}\times$$

$$\times\sum_{k=0}^{n}\frac{\theta(aq^{2k};p)\left[a,b,c,\frac{a}{bc};q,p\right]_k\left[q^{-n},\frac{\beta}{\alpha}q^{-n},\frac{\gamma}{\alpha}q^{-n},\frac{q^{-n}}{\beta\gamma};q,p\right]_k q^k}{\theta(a;p)\left[q,\frac{aq}{b},\frac{aq}{c},bcq;q,p\right]_k\left[\frac{q^{-n}}{\alpha},\frac{q^{-n}}{\beta},\frac{q^{-n}}{\gamma},\frac{\beta\gamma q^{-n}}{\alpha};q,p\right]_k}$$

$$=\frac{\left[aq,\frac{aq}{bc},bq,cq;q,p\right]_n}{\left[q,\frac{aq}{b},\frac{aq}{c},bcq;q,p\right]_n}\times$$

$$\times\sum_{k=0}^{n}\frac{\theta(\alpha q^{2k};p)\left[\alpha,\beta,\gamma,\frac{\alpha}{\beta\gamma};q,p\right]_k\left[q^{-n},\frac{b}{a}q^{-n},\frac{c}{a}q^{-n},\frac{q^{-n}}{bc};q,p\right]_k q^k}{\theta(\alpha;p)\left[q,\frac{\alpha q}{\beta},\frac{\alpha q}{\gamma},\beta\gamma q;q,p\right]_k\left[\frac{q^{-n}}{a},\frac{q^{-n}}{b},\frac{q^{-n}}{c},\frac{bcq^{-n}}{a};q,p\right]_k} \tag{2.13}$$

(vii) Next setting

$$\alpha_r = \frac{\theta(aq^{2r};p)[a,b,c,a/bc;q,p]_r q^r}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_r}$$

and

$$\delta_r = \frac{\theta(\alpha p_1 q_1^{2r};p_1^2)[\alpha p_1,\beta,\gamma,\alpha p_1/\beta\gamma;q_1,p_1^2]_r q_1^r}{\theta(\alpha p_1;p_1^2)[q_1,\alpha p_1 q_1/\beta,\alpha p_1 q_1/\gamma,\beta\gamma q_1;q_1,p_1^2]_r}$$

in (1.24) and using (1.11) and (1.14), we get

$$ {}_{12}E_{11}\left[\begin{matrix} \alpha p_1, q_1\sqrt{\alpha p_1}, -q_1\sqrt{\alpha p_1}, q_1\sqrt{\alpha/p_1}, -q_1\sqrt{\alpha p_1^3}, \beta,\gamma,\alpha p_1/\beta\gamma; \\ \sqrt{\alpha p_1}, -\sqrt{\alpha p_1}, \sqrt{\alpha p_1^3}, -\sqrt{\alpha/p_1}, \alpha p_1 q_1/\beta, \alpha p_1 q_1/\gamma, \beta\gamma q_1; \end{matrix}\right.$$

$$\left.\begin{matrix} aq, bq, cq, aq/bc; (q_1,p_1^2),(q,p); -1 \\ ; q, aq/b, aq/c, bcq \end{matrix}\right]_n =$$

$$= \frac{[aq,bq,cq,aq/bc;q,p]_n[\alpha p_1 q_1,\alpha p_1 q_1/\beta\gamma,\beta q_1,\gamma q_1;q_1,p_1^2]_n}{[q,aq/b,aq/c,bcq;q,p]_n[\alpha p_1 q_1/\beta,\alpha q_1 p_1/\gamma,\beta\gamma q_1,q_1;q_1,p_1^2]_n}$$

$$\times - \frac{\theta(aq^2,b,c,a/bc;p)}{\theta(q,aq/b,aq/c,bcq;p)}\ {}_{12}E_{11}\left[\begin{matrix} \alpha p_1 q_1, \alpha p_1 q_1/\beta\gamma, \beta q_1, \gamma q_1; \\ \alpha p_1 q_1/\beta, \alpha p_1 q_1/\gamma, \beta\gamma q_1; \end{matrix}\right.$$

$$\left.\begin{matrix} ; aq^2, q^2\sqrt{a}, -q^2\sqrt{a}, q^2\sqrt{a/p}, q^2\sqrt{ap}, bq, cq, aq/bc; (q_1,p_1^2),(q,p); -1 \\ ; q^2, q\sqrt{a}, -q\sqrt{a}, q\sqrt{ap}, -q\sqrt{a/p}, aq^2/b, aq^2/c, bcq^2 \end{matrix}\right]_{n-1}$$

which a transformation involving truncated series.

(viii) Setting

$$\alpha_r = \frac{\theta(aq^{2r};p)[a,b,c,a/bc;q,p]_r q^r}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_r}$$

and

$$\delta_k = \frac{\theta\{\alpha(rst/q_1)^k,\beta(r/q_1)^k,\gamma(s/q_1)^k,(\alpha/\beta\gamma)(t/q_1)^k;p\}}{\theta(\alpha,\beta,\gamma,\alpha/\beta\gamma;p)}$$

$$\times \frac{[\alpha;rst/q_1^2,p_1]_k[\beta;r,p_1]_k[\gamma;s,p_1]_k[\alpha/\beta\gamma;t,p_1]_k q_1^k}{[q_1;q_1,p_1]_k[\alpha st/\beta q_1;st/q_1,p_1]_k[art/\gamma q_1;rt/q_1,p_1]_k[\beta\gamma rs/q_1;sr/q_1,p_1]_k}$$

in (1.24) and using (1.11) and (1.16), we get

$$\sum_{m=0}^{n} \frac{\theta\{\alpha(rst/q_1)^m,\beta(r/q_1)^m,\gamma(s/q_1)^m,(\alpha/\beta\gamma)(t/q_1)^m;p\}}{\theta(\alpha,\beta,\gamma,\alpha/\beta\gamma;p)}$$

$$\times \frac{[\alpha;rst/q_1^2,p_1]_m[\beta;r,p_1]_m[\gamma;s,p_1]_m[\alpha/\beta\gamma;t,p_1]_m q_1^m}{[q_1;q_1,p_1]_m[\alpha st/\beta q_1;st/q_1,p_1]_m[\alpha rt/\gamma q_1;rt/q_1,p_1]_m[\beta\gamma rs/q_1;sr/q_1,p_1]_m}$$

$$\times \frac{[aq, bq, cq, aq/bc; q, p]_m}{[q, aq/b, aq/c, bcq; q, p]_m}$$

$$= \frac{[aq, bq, cq, aq/bc; q, p]_n [\alpha rst/q_1^2, p_1]_n [\beta r; r, p_1]_n [\gamma s; s, p_1]_n}{[q, aq/b, aq/c, bcq; q, p]_n [q_1; q_1, p_1]_n [\alpha st/\beta q_1; st/q_1, p_1]_n [\alpha rt/\gamma q_1; rt/q_1, p_1]_n}$$

$$\frac{[\alpha t/\beta\gamma; t, p_1]_n}{[\beta\gamma rs/q_1; sr/q_1, p_1]_n} - \frac{q\theta(aq^2, b, c, a/bc; p)}{\theta(q, aq/b, aq/c, bcq; p)} \times$$

$$\sum_{k=0}^{n-1} \frac{\theta(aq^{2k+2}; p)[aq, bq, cq, aq/bc; q, p]_k [\alpha rst/q_1^2; rst/q_1^2, p_1]_k}{\theta(aq^2; p)[q^2, aq^2/b, aq^2/c, bcq^2; q, p]_k [q_1; q_1, p_1]_k} \times$$

$$\times \frac{[\beta r; r, p_1]_k [\gamma s; s, p_1]_k [\alpha t/\beta\gamma; t, p_1]_k q^k}{[\alpha st/\beta q_1; st/q_1, p_1]_k [\alpha rt/\gamma q_1; rt/q_1, p_1]_k [\beta\gamma rs/q_1; sr/q_1, p_1]_k} \tag{2.14}$$

For $r = s = t = q_1 = q$ and $p_1 = p$, (2.14) yields

$$\sum_{m=0}^{n} \frac{\theta(\alpha q^{2m}; p) \left[\alpha, \beta, \gamma, \frac{\alpha}{\beta\gamma}; q, p\right]_m}{\theta(\alpha; p) \left[q, \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \beta\gamma q; q, p\right]_m} \frac{\left[aq, \frac{aq}{bc}, bq, cq; q, p\right]_m}{\left[q, \frac{aq}{b}, \frac{aq}{c}, bcq; q, p\right]_m}$$

$$= \frac{\left[aq, bq, cq, \frac{aq}{bc}, \alpha q, \beta q, \gamma q, \frac{\alpha q}{\beta\gamma}; q, p\right]_n}{\left[q, q, \frac{aq}{b}, \frac{aq}{c}, bcq, \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \beta\gamma q; q, p\right]_n} - \frac{q\theta\left[aq^2, b, c, \frac{a}{bc}; p\right]}{\theta\left[q, \frac{aq}{b}, \frac{aq}{c}, bcq; p\right]} \times$$

$$\times \sum_{k=0}^{n-1} \frac{\theta(aq^{2k+2}; p) \left[aq, bq, cq, \frac{aq}{bc}, \alpha q, \beta q, \gamma q, \frac{\alpha q}{\beta\gamma}; q, p\right]_k q^k}{\theta(aq^2; p) \left[q^2, \frac{aq^2}{b}, \frac{aq^2}{c}, bcq^2, q, \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \beta\gamma q; q, p\right]_k}. \tag{2.15}$$

(ix) Choosing

$$\alpha_k = \frac{\theta(apq^{2k}; p^2) \left[ap, b, c, \frac{ap}{bc}; q, p^2\right]_k q^k}{\theta(ap; p^2) \left[q, \frac{apq}{b}, \frac{apq}{c}, bcq; q, p^2\right]_k}$$

and

$$\delta_k = \frac{\theta\left\{\alpha\left(\frac{rst}{q_1}\right)^k, \beta\left(\frac{r}{q_1}\right)^k, \gamma\left(\frac{s}{q_1}\right)^k, \frac{\alpha}{\beta\gamma}\left(\frac{t}{q_1}\right)^k; p_1\right\}}{\theta\left(\alpha, \beta, \gamma, \frac{\alpha}{\beta\gamma}; p_1\right)}$$

$$\times \frac{\left(\alpha; \frac{rst}{q_1^2}, p_1\right)_k (\beta; r, p_1)_k (\gamma; s, p_1)_k \left(\frac{\alpha}{\beta\gamma}; t, p_1\right)_k q_1^k}{(q_1; q_1, p_1)_k \left(\frac{\alpha}{\beta}\frac{st}{q_1}; \frac{st}{q_1}, p_1\right)_k \left(\frac{\alpha}{\gamma}\frac{rt}{q_1}; \frac{rt}{q_1}, p_1\right)_k \left(\beta\gamma\frac{rs}{q_1}; \frac{rs}{q_1}, p_1\right)_k}$$

in (1.24) and using (1.14) and (1.16) we find

$$\sum_{k=0}^{n} \frac{\theta\left\{\alpha\left(\frac{rst}{q_1}\right)^k, \beta\left(\frac{r}{q_1}\right)^k, \gamma\left(\frac{s}{q_1}\right)^k, \frac{\alpha}{\beta\gamma}\left(\frac{t}{q_1}\right)^k; p_1\right\}}{\theta\left(\alpha, \beta, \gamma, \frac{\alpha}{\beta\gamma}; p_1\right)}$$

$$\times \frac{\left(\alpha; \frac{rst}{q_1^2}, p_1\right)_k (\beta; r, p_1)_k (\gamma; s, p_1)_k \left(\frac{\alpha}{\beta\gamma}; t, p_1\right)_k q_1^k}{(q_1; q_1, p_1)_k \left(\frac{\alpha}{\beta}\frac{st}{q_1}; \frac{st}{q_1}, p_1\right)_k \left(\frac{\alpha}{\gamma}\frac{rt}{q_1}; \frac{rt}{q_1}, p_1\right)_k \left(\beta\gamma\frac{rs}{q_1}; \frac{rs}{q_1}, p_1\right)_k} \times$$

$$\times \frac{\left[apq, bq, cq, \frac{apq}{bc}; q, p^2\right]_n}{\left[q, \frac{apq}{b}, \frac{apq}{c}, bcq; q, p^2\right]_n}$$

$$= \frac{\left[apq, bq, cq, \frac{apq}{bc}; q, p^2\right]_n \left(\alpha\frac{rst}{q_1^2}; \frac{rst}{q_1^2}, p_1\right)_n (\beta r; r, p_1)_n}{\left[q, \frac{apq}{b}, \frac{apq}{c}, bcq; q, p^2\right]_n (q_1; q_1, p_1)_n \left(\frac{\alpha}{\beta}\frac{st}{q_1}; \frac{st}{q_1}, p_1\right)_n} \times$$

$$\times \frac{(\gamma s; s, p_1)_n \left(\frac{\alpha t}{\beta\gamma}; t, p_1\right)_n}{\left(\frac{\alpha}{\gamma}\frac{rt}{q_1}; \frac{rt}{q_1}, p_1\right)_n \left(\beta\gamma\frac{rs}{q_1}; \frac{rs}{q_1}, p_1\right)_n} - \frac{q\theta\left[apq^2, b, c, \frac{ap}{bc}; p^2\right]}{\theta\left[q, \frac{apq}{b}, \frac{apq}{c}, bcq; p^2\right]}$$

$$\times \sum_{k=0}^{n-1} \frac{\theta(apq^{2k+2}; p^2)\left[apq, b, c, \frac{apq}{bc}; q, p^2\right]_k q^k}{\theta(apq^2; p^2)\left[q^2, \frac{apq^2}{b}, \frac{apq^2}{c}, bcq^2; q, p^2\right]_k} \times$$

$$\times \frac{\left(\alpha\frac{rst}{q_1^2}; \frac{rst}{q_1^2}, p_1\right)_k (\beta r; r, p_1)_k}{(q_1; q_1, p_1)_k \left(\frac{\alpha}{\beta}\frac{st}{q_1}; \frac{st}{q_1}, p_1\right)_k} \frac{(\gamma s; s, p_1)_k \left(\frac{\alpha t}{\beta\gamma}; t, p_1\right)_k}{\left(\frac{\alpha}{\gamma}\frac{rt}{q_1}; \frac{rt}{q_1}, p_1\right)_k \left(\beta\gamma\frac{rs}{q_1}; \frac{rs}{q_1}, p_1\right)_k}. \tag{2.16}$$

For $r = s = t = q_1 = q$ and $p_1 = p$, (2.16) yields

$$\sum_{k=0}^{n} \frac{\theta(\alpha q^{2k};p)\left[\alpha,\beta,\gamma,\frac{\alpha}{\beta\gamma};q,p\right]_k q^k}{\theta(\alpha;p)\left[q,\frac{\alpha q}{\beta},\frac{\alpha q}{\gamma},\beta\gamma q;q,p\right]_k} \frac{\left[apq,\frac{apq}{bc},bq,cq;q,p^2\right]_k}{\left[q,\frac{apq}{b},\frac{apq}{c},bcq;q,p^2\right]_k}$$

$$= \frac{\left[apq,bq,cq,\frac{apq}{bc};q,p^2\right]_n \left[\alpha q,\beta q,\gamma q,\frac{\alpha q}{\beta\gamma};q,p\right]_n}{\left[q,\frac{apq}{b},\frac{apq}{c},bcq;q,p^2\right]_n \left[q,\frac{\alpha q}{\beta},\frac{\alpha q}{\gamma},\beta\gamma q;q,p\right]_n} - \frac{q\theta\left[ap,b,c,\frac{ap}{bc};p^2\right]}{\theta\left[q,\frac{apq}{b},\frac{apq}{c},bcq;p^2\right]}\times$$

$$\times\sum_{k=0}^{n-1} \frac{\theta(apq^{2k+2};p^2)\left[apq,bq,cq,\frac{apq}{bc};q,p^2\right]_k q^k \left[\alpha q,\beta q,\gamma q,\frac{\alpha q}{\beta\gamma};q,p\right]_k}{\theta(ap;p^2)\left[q^2,\frac{apq^2}{b},\frac{apq^2}{c},bcq^2;q,p^2\right]_k \left[q,\frac{\alpha q}{\beta},\frac{\alpha q}{\gamma},\beta\gamma q;q,p\right]_k}. \qquad (2.17)$$

It is evident that we can establish several other interesting transformations involving theta functions.

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