J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.6, No.2 (2017), pp. 31-44

THE BACKWARD OPERATOR OF DOUBLE ALMOST $(\lambda_m \mu_n)$ CONVERGENCE IN χ^2-RIESZ SPACE DEFINED BY A MUSIELAK-ORLICZ FUNCTION

Vandana[1], Deepmala[2], N. Subramanian[3] & Lakshmi Narayan Mishra[4]*

[1]Department of Management Studies,
Indian Institute of Technology, Madras, Tamil Nadu-600 036, INDIA
E-mail: vdrai1988@gmail.com

[2]Mathematics Discipline,
PDPM Indian Institute of Information Technology,
Design and Manufacturing, Jabalpur-482 005, INDIA
E-mail: dmrai23@gmail.com

[3]Department of Mathematics,
SASTRA University, Thanjavur-613 401, INDIA
E-mail: nsmaths@yahoo.com

[4]*Department of Mathematics,
Lovely Professional University, Phagwara, Punjab-144 411, INDIA
E-mail: lakshminarayanmishra04@gmail.com
*Corresponding Author

Abstract: In this paper we introduce the backward operator is ∇ and study the notion of $\nabla-$ statistical convergence and $\nabla-$ statistical Cauchy sequence using by almost $(\lambda_m \mu_n)$ convergence in χ^2-Riesz space and also some inclusion theorems are discussed.

Keywords: Analytic sequence, Museialk-Orlicz function, double sequences, chi sequence,Lambda, Riesz space, strongly, statistical convergent.

2010 Mathematics Subject Classification: 40A05, 40C05, 40D05.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Tripathy [1] and Mursaleen [2] and Mursaleen and Edely [3,4], Subramanian and Misra [5], Pringsheim [6], Moricz and Rhoades [7], Robison [8], Savas et al. [9], Raj et al. [10], Francesco Tulone [11] and many others.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...).$$

A double sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x, y) = sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m, n : 1, 2, 3, ... \right\}, \tag{1.1}$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ $in\,\Gamma^2$. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$$

with 1 in the $(m, n)^{th}$ position and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if $((m + n)! |x_{mn}|)^{\frac{1}{m+n}} \to$

0 as $m, n \to \infty$. The double gai sequences will be denoted by χ^2.

2. Definitions and Preliminaries

A double sequence $x = (x_{mn})$ has limit 0 (denoted by $P - limx = 0$) (i.e) $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. We shall write more briefly as $P - convergent\ to\ 0$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function. An Orlicz function M is said to satisfy Δ_2- condition for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

2.1. Lemma. Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.

A double sequence $M = (M_{mn})$ of Orlicz function is called a Musielak-Orlicz function [see [12]]. A double sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup \{|v| u - (M_{mn})(u) : u \geq 0\}, m, n = 1, 2, \cdots$$

is called the complementary function of a sequence of Musielak-Orlicz M. For a given sequence of Musielak-Orlicz function M, the Musielak-Orlicz sequence space t_M is defined as follows

$$t_M = \left\{x \in w^2 : I_M(|x_{mn}|)^{1/m+n} \to 0\, \text{as}\, m, n, k \to \infty\right\},$$

where I_M is a convex modular defined by

$$I_M(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|)^{1/m+n}.$$

2.2. Definition. A double sequence $x = (x_{mn})$ of real numbers is called almost $P-$ convergent to a limit 0 if

$$P - lim_{p,q \to \infty} sup_{r,s \geq 0} \tfrac{1}{pq} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} ((m+n)! |x_{mn}|)^{1/m+n} \to 0.$$

that is, the average value of (x_{mn}) taken over any rectangle $\{(m, n) : r \leq m \leq r+p-1, s \leq n \leq s+q-1\}$ tends to 0 as both p and q to ∞, and this $P-$ convergence is uniform in r and s. Let denote the set of sequences with this property as $\left[\widehat{\chi^2}\right]$.

2.3. Definition. Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and
$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1, \ \mu_{n+1} \leq \mu_n + 1, \mu_1 = 1.$
Let $I_m = [m - \lambda_m + 1, m]$ and $I_n = [n - \mu_n + 1, n]$.
For any set $K \subseteq \mathbb{N} \times \mathbb{N}$, the number
$\delta_{\lambda,\mu}(K) = lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} \left|\{(i,j) : i \in I_m, j \in I_n, (i,j) \in K\}\right|$, is called the $(\lambda, \mu)-$ density of the set K provided the limit exists. [See [31]].

2.4. Definition. A double sequence $x = (x_{mn})$ of numbers is said to be $(\lambda, \mu)-$ statistical convergent to a number ξ provided that for each $\epsilon > 0$,
$lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} \left|\{(i,j) : i \in I_m, j \in I_n, |x_{mn} - \xi| \geq \epsilon\}\right| = 0,$
that is, the set $K(\epsilon) = \frac{1}{\lambda_m \mu_n} \left|\{(i,j) : i \in I_m, j \in I_n, |x_{mn} - \xi| \geq \epsilon\}\right|$ has $(\lambda, \mu)-$ density zero. In this case the number ξ is called the $(\lambda, \mu)-$ statistical limit of the sequence $x = (x_{mn})$ and we write $St_{(\lambda,\mu)} lim_{m,n\to\infty} = \xi$.

2.5. Definition. Let M be an Orlicz function and $P = (p_{mn})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence space: $\chi^2_M [AC_{\lambda_m \mu_n}, P] =$
$\left\{P - lim_{m,n} \frac{1}{\lambda_m \mu_n} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} \left[M\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} = 0, \right\}$,
uniformly in r and s.

We shall denote $\chi^2_M [AC_{\lambda_m \mu_n}, P]$ as $\chi^2 [AC_{\lambda_m \mu_n}]$ respectively when $p_{mn} = 1$ for all m and n. If x is in $\chi^2 [AC_{\lambda_m \mu_n}, P]$, we shall say that x is almost $(\lambda_m \mu_n)$ in χ^2 strongly $P-$convergent with respect to the Orlicz function M. Also note if $M(x) = x, p_{mn} = 1$ for all m, n and k then $\chi^2_M [AC_{\lambda_m \mu_n}, P] = \chi^2 [AC_{\lambda_m \mu_n}, P]$, which are defined as follows: $\chi^2 [AC_{\lambda_m \mu_n}, P] =$
$\left\{P - lim_{m,n} \frac{1}{\lambda_m \mu_n} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} \left[M\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right] = 0, \right\}$, uniformly in r and s.
Again note if $p_{mn} = 1$ for all m and n then $\chi^2_M [AC_{\lambda_m \mu_n}, P] = \chi^2_M [AC_{\lambda_m \mu_n}]$. We define $\chi^2_M [AC_{\lambda_m \mu_n}, P] =$
$\left\{P - lim_{m,n} \frac{1}{\lambda_m \mu_n} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} \left[M\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} = 0, \right\}$,
uniformly in r and s.

2.6. Definition. Let M be an Orlicz function and $P = (p_{mn})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence space: $\chi^2_M [P] =$
$\left\{P - lim_{p,q\to\infty} \frac{1}{pq} \sum_{m=1}^{p} \sum_{n=1}^{q} \left[M\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} = 0\right\}$, uniformly in r and s.

If we take $M(x) = x, p_{mn} = 1$ for all m and n then $\chi^2_M [P] = \chi^2$.

2.7. Definition. The double number sequence x is $\widehat{S_{\lambda_m \mu_n}} - P-$ convergent to 0 then
$P - lim_{m,n} \frac{1}{\lambda_m \mu_n} max_{r,s} \left| \left\{ (m,n) \in I_{r,s} : M\left((m+n)! \left| x_{m+r,n+s} - 0 \right| \right)^{1/m+n} \right\} \right| = 0.$
In this case we write $\widehat{S_{\lambda_m \mu_n}} - lim \left(M \left(m+n\right)! \left| x_{m+r,n+s} - 0 \right| \right)^{1/m+n} = 0.$

3. The Backward operator of convergence of double almost $(\lambda_m \mu_n)$ in χ^2 Riesz space

Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$ $(m\, be\, infinite)$, τ a triangle, and $F : (X \times X) \times (X \times X) \rightarrow D^+$. Then F is called a probabilistic Riesz space. A real valued function $F\left(d_p(x_1, \ldots, x_n), t\right) = F\left(\|(d_1(x_1,0), \ldots, d_n(x_n,0))\|_p, t\right)$ on X satisfying the following four conditions:
(i) $F\left(\|(d_1(x_1,0), \ldots, d_n(x_n,0))\|_p, t\right) = 0$ if and and only if $F(d_1(x_1,0), \ldots, d_n(x_n, 0), t)$ are linearly dependent,
(ii) $F\left(\|(d_1(x_1,0), \ldots, d_n(x_n,0))\|_p, t\right)$ is invariant under permutation,
(iii) $F\left(\|(\alpha d_1(x_1,0), \ldots, \alpha d_n(x_n,0))\|_p, t\right) = F\left(|\alpha| \, \|(d_1(x_1,0), \ldots, d_n(x_n,0))\|_p, t\right),$ $\alpha \in \mathbb{R}$
(iv) $F(d_p((x_1,y_1),(x_2,y_2) \cdots (x_n,y_n),t) = F(d_X(x_1,x_2, \cdots x_n)^p, t) + F(d_Y \ (y_1, y_2, \cdots y_n)^p \ )^{1/p}, t)$ for $1 \leq p < \infty$; (or)
(v) $F(d((x_1,y_1),(x_2,y_2), \cdots (x_n,y_n)),t) := \sup F(\{d_X(x_1,x_2, \cdots x_n), d_Y \ (y_1, y_2, \cdots y_n) \ \}, t)$, for $(x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y, F, *)$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$F\left(\|(d_1(x_1,0), \ldots, d_n(x_n,0))\|_E, t\right) = sup\, F\left(|det(d_{mn}\left(x_{mn},0\right))|, t\right) =$$
$$sup \left(\begin{vmatrix} d_{11}\left(x_{11},0\right) & d_{12}\left(x_{12},0\right) & \ldots & d_{1n}\left(x_{1n},0\right) \\ d_{21}\left(x_{21},0\right) & d_{22}\left(x_{22},0\right) & \ldots & d_{2n}\left(x_{1n},0\right) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}\left(x_{n1},0\right) & d_{n2}\left(x_{n2},0\right) & \ldots & d_{nn}\left(x_{nn},0\right) \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

3.1. Definition. Let L be a real vector space and let $\leq$ be a partial order on this space. L is said to be an ordered vector space if it satisfies the following properties :

(i) If $x, y \in L$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in L$.

(ii) If $x, y \in L$ and $y \leq x$, then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.

If in addition L is a lattice with respect to the partial ordering, then L is said to be Riesz space.

A subset S of a Riesz space X is said to be solid if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.

A linear topology τ on a Riesz space X is said to be locally solid if τ has a base at zero consisting of solid sets.

3.2. Definition. Let $\chi^{2\tau}_M \left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) of points in χ^2 is said to be $S(\tau)-$ convergent in $(X, F, *)$ if for each $t > 0$, $\theta \in (0,1)$ and for non zero $z \in X$ such that

$$\delta\left(\left\{m,n \in \mathbb{N} : F\left(M_{mn}\left(((m+n)!\,|x_{mn}|)^{1/m+n}, z; t\right) \leq 1-\theta\right\}\right)\right) = 0$$

that is , $\Big(P - lim_{m,n} \frac{1}{\lambda_m \mu_n} \Big\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F$
$\left(\left[M\left((m+n)!\,|x_{m+r,n+s}| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right) \leq 1-\theta\Big\}\Big) = 0.$

In this case we write

$S(\tau) - \Big(P - lim_{m,n} \frac{1}{\lambda_m \mu_n} \Big\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}}$
$F\left(\left[M\left((m+n)!\,|x_{m,n}| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\Big\}\Big) = 1.$

3.3. Definition. Let $\chi^{2\tau}_M \left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) of points in χ^2 is said to be $\nabla-$ convergent in $(X, F, *)$ if for each $t > 0$, $\beta \in (0,1)$ there exists an positive integer n_0 such that

$$F\left(M_{mn}\left(((m+n)!\,|x_{mn}|)^{1/m+n}, z; t\right)\right\} > 1-\beta.$$

whenever $m, n \geq n_0$ and for non zero $z \in X$.

3.4. Definition. Let $\chi^{2\tau}_M \left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) of points in χ^2 is said to be $\nabla-$ Cauchy in $(X, F, *)$ if for each $t > 0$, $\beta \in (0,1)$ there exists an positive integer $n_0 = n_0(\epsilon)$ such that

$$F\left(M_{mn}\left(((m+n)!\left|x_{mn}-x_{rs}\right|)^{1/m+n}, z; t\right)\right\} < 1-\theta.$$

whenever $m, n, r, s \geq n_0$ and for non zero $z \in X$.

3.5. Definition. Let $\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \left\|\left(d\left(x_1, 0\right), d\left(x_2, 0\right), \cdots, d\left(x_{n-1}, 0\right)\right)\right\|_p\right]$ be a Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) of points in χ^2 is said to be $S(\tau)-$ convergent in $(X, F, *)$ if for each $t > 0$, $\beta \in (0,1)$ and for non zero $z \in X$ such that

$$\delta_\nabla\left(\left\{m, n \in \mathbb{N} : F\left(M_{mn}\left(((m+n)!\left|x_{mn}\right|)^{1/m+n}, z; t\right) \leq 1-\beta\right\}\right)\right) = 0$$

that is ,
$\left(P - lim_{m,n}\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m+r,n+s}\right|-0\right)^{1/m+n}\right]^{p_{mn}}\right.\right.\right.$
$, z; t) \leq 1-\beta\}\ ) = 0.$
In this case we write
$S(\tau)_\nabla - \left(P - lim_{m,n}\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}}\right.\right.\right.$
$, z; t)\ \}\ ) = 1.$

3.6. Definition. Let $\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \left\|\left(d\left(x_1, 0\right), d\left(x_2, 0\right), \cdots, d\left(x_{n-1}, 0\right)\right)\right\|_p\right]$ be a Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) of points in χ^2 is said to be $\nabla-$ Cauchy in $(X, F, *)$ if for each $t > 0$, $\beta \in (0,1)$ there exists an positive integer $n_0 = n_0(\epsilon)$ such that

$$\delta_\nabla\left(\left\{m, n \in \mathbb{N} : F\left(M_{mn}\left(((m+n)!\left|x_{mn}-x_{rs}\right|)^{1/m+n}, z; t\right) \leq 1-\beta\right\}\right)\right) = 0$$

or equivalently,

$$\delta_\nabla\left(\left\{m, n \in \mathbb{N} : F\left(M_{mn}\left(((m+n)!\left|x_{mn}-x_{rs}\right|)^{1/m+n}, z; t\right) > 1-\beta\right\}\right)\right) = 1$$

4. Main Results

4.1. Proposition. Let $\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \left\|\left(d\left(x_1, 0\right), d\left(x_2, 0\right), \cdots, d\left(x_{n-1}, 0\right)\right)\right\|_p\right]$ be a Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) of χ^2 in $(X, F, *)$ if for each $t > 0$, $\beta \in (0,1)$ and for non zero $z \in X$, then the following statements are equivalent

(i)$\delta_\nabla\left(\left\{m, n \in \mathbb{N} : F\left(M_{mn}\left(((m+n)!\left|x_{mn}-x_{rs}\right|)^{1/m+n}, z; t\right) \leq 1-\beta\right\}\right)\right) = 0$
(ii)$\delta_\nabla\left(\left\{m, n \in \mathbb{N} : F\left(M_{mn}\left(((m+n)!\left|x_{mn}-x_{rs}\right|)^{1/m+n}, z; t\right) > 1-\beta\right\}\right)\right) = 1$
(iii)$S(\tau)_\nabla - \left(P - lim_{m,n}\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}}\right.\right.\right.$

$, z; t) \}) = 1.$

4.2. Theorem. Let $\chi^{2\tau}_{M}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \in S(\tau)_{\nabla}$ and $c \in \mathbb{R}$ be a almost $(\lambda_m \mu_n)$ Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) in $(X, F, *)$ then

(i)$S(\tau)_{\nabla} - \left(P - clim_{m,n} \frac{1}{\lambda_m \mu_n} \left\{ \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right| - 0\right)^{1/m+n}\right]^{p_{mn}}\right.\right.\right.$
$\left.\left.\left., z; t\right)\right\}\right) = c\, S(\tau)_{\nabla} - \left(P - lim_{m,n} \frac{1}{\lambda_m \mu_n} \left\{ \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|\right.\right.\right.\right.\right.$
$\left.\left.\left.\left.\left. -0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right)$

(ii)

$S(\tau)_{\nabla} - \left(P - lim_{m,n} \frac{1}{\lambda_m \mu_n} \left\{ \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n} + y_{mn}\right|\right.\right.\right.\right.\right.$
$\left.\left.\left.\left.\left. -0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right) = S(\tau)_{\nabla} - \left(P - lim_{m,n} \frac{1}{\lambda_m \mu_n} \left\{ \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}}\right.\right.$
$\left.\left. F\left(\left[M\left((m+n)!\left|x_{m,n}\right| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right) +$
$S(\tau)_{\nabla} - \left(P - lim_{m,n} \frac{1}{\lambda_m \mu_n} \left\{ \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|y_{m,n}\right|\right.\right.\right.\right.\right.$
$\left.\left.\left.\left.\left. -0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right)$

Proof: The proof of this theorem is straightforward, and thus will be omitted.

4.3. Theorem. Let $\chi^{2\tau}_{M}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a almost $(\lambda_m \mu_n)$ Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) analytic in $(X, F, *)$ then

(a)$\chi^{2\tau}_{M}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to W(\tau)_{\nabla}$ implies
$\chi^{2\tau}_{M}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to S(\tau)_{\nabla}.$

(b)$\Lambda^{2\tau}_{M}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to S(\tau)_{\nabla}$ imply
$\Lambda^{2\tau}_{M}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to W(\tau)_{\nabla}.$

(c)$S(\tau)_{\nabla} \bigcap \Lambda^{2\tau}_{M}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$W(\tau)_{\nabla} \bigcap \Lambda^{2\tau}_{M}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right].$

Proof: Let $\epsilon > 0$ and $\chi^{2\tau}_{M}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to W(\tau)_{\nabla}$ for all $r, s \in \mathbb{N}$, we have

$$\left(lim_{m,n} \frac{1}{\lambda_m \mu_n} \left\{ \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right) \geq \epsilon$$

$\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right\} \geq$
$\left|\left(lim_{m,n}\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\geq\epsilon\right\}\right)\right|\cdot$
$min\left(\epsilon^h,\epsilon^H\right).$
Hence $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]\to S\left(\tau\right)_\nabla.$

Proof(b): Suppose that $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]\in S\left(\tau\right)_\nabla\bigcap\Lambda^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$. Since $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]\in$ $\Lambda^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$, we write $\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right\}\leq T$, for all $r,s\in\mathbb{N}$, let
$G_{rs}=\left|\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\geq\epsilon\right\}\right)\right|$
and
$H_{rs}=\left|\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)<\epsilon\right\}\right)\right|.$
Then we have
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right\}\right)=$
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in G_{r,s}}\sum_{n\in G_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right\}\right)+$
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in H_{r,s}}\sum_{n\in H_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right\}\right)\leq$
$max\left(T^h,T^H\right)G_{rs}+max\left(\epsilon^h,\epsilon^H\right)$. Taking the limit as $\epsilon\to 0$ and $r,s\to\infty$, it follows that $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]\in W\left(\tau\right)_\nabla.$
(c) Follows from (a) and (b).

4.4. Theorem. If $liminf_{rs}\left(\frac{\lambda_r\mu_s}{rs}\right)>0$, then $S\left(\tau\right)\subset S\left(\tau\right)_\nabla$

Proof: Let $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]\in S\left(\tau\right)$. For given $\epsilon>0$, we get
$\left|\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}}\right)\geq\epsilon\right\}\right)\right|\supset G_{rs}$ where G_{rs} is in the theorem of 4.3.(b). Thus,
$\left|\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\geq\epsilon\right\}\right)\right|\geq G_{rs}=$
$\frac{\lambda_r\mu_s}{rs}$. Taking limit as $r,s\to\infty$ and using $liminf_{rs}\left(\frac{\lambda_r\mu_s}{rs}\right)>0$, we get
$\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]\in S\left(\tau\right)_\nabla.$

4.5. Theorem. Let $0 < u_{mn} \leq v_{mn}$ and $(u_{mn} v_{mn}^{-1})$ be double analytic. Then $W(\tau, v)_{\nabla} \subset w(\tau, u)_{\nabla}$

Proof: Let $\chi_M^{2\tau}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \in W(\tau, v)_{\nabla}$.
Let $W(\tau)_{\nabla} = \left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\,|x_{m,n}| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right)$ for all $r, s \in \mathbb{N}$ and $\lambda_m \mu_n = u_{mn} v_{mn}^{-1}$ for all $m, n \in \mathbb{N}$. Then $0 < \lambda_m \mu_n \leq 1$ for all $m, n \in \mathbb{N}$. Let b be a constant such that $0 < b \leq \lambda_m \mu_n \leq 1$ for all $m, n \in \mathbb{N}$.

Define the double sequences (k_{mn}) and (ℓ_{mn}) as follows:
For $W(\tau)_{\nabla} \geq 1$, let $(k_{mn}) = (W(\tau)_{\nabla})$ and $\ell_{mn} = 0$ and for $W(\tau)_{\nabla} < 1$, let $k_{mn} = 0$ and $\ell_{mn} = W(\tau)_{\nabla}$. Then it is clear that for all $m, n \in \mathbb{N}$, we have $W(\tau)_{\nabla} = k_{mn} + \ell_{mn}$ and $W(\tau)_{\nabla}^{\lambda_m \mu_n} = k_{mn}^{\lambda_m \mu_n} + \ell_{mn}^{\lambda_m \mu_n}$. Now it follows that $k_{mn}^{\lambda_m \mu_n} \leq k_{mn} \leq W(\tau)_{\nabla}$ and $\ell_{mn}^{\lambda_m \mu_n} \leq \ell_{mn}^{\lambda \mu}$. Therefore

$$\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|W(\tau)_{\nabla}^{\lambda_m \mu_n}\right| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right) =$$
$$\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|(k_{mn} + \ell_{mn})^{\lambda_m \mu_n}\right| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right) =$$
$$\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|W(\tau)_{\nabla}^{\lambda_m \mu_n}\right| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right) +$$
$$\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|(\ell_{mn})^{\lambda_m \mu_n}\right| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right).$$

Now for each r, s,

$$\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|(\ell_{mn})^{\lambda \mu}\right| - 0\right)^{1/m+n}\right]^{p_{mn}}, z; t\right)\right\}\right) =$$
$$\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|\left((\ell_{mn})^{\lambda \mu}\left(\frac{1}{\lambda_m \mu_n}\right)^{1-\lambda \mu}\right)\right| - 0\right)^{1/m+n}\right]^{p_{mn}}\right.\right.\right.$$
$$\left., z; t)\}\right)$$
$$\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\left|\left(\left((\ell_{mn})^{\lambda \mu}\right)^{\lambda \mu}\right)^{1/\lambda \mu}\right| - 0\right)^{1/m+n}\right]^{p_{mn}}\right.\right.\right.$$
$$\left., z; t)\}\right)^{\lambda \mu}$$

4.6. Theorem.
$\Lambda_M^{2\tau}\left[AC_{\lambda_m \mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = W(\tau, \Lambda^2)_{\nabla}$, where $W(\tau, \Lambda^2)_{\nabla} = sup\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} F\left(\left[M\left((m+n)!\,|x_{m,n}| - 0\right)^{1/m+n}\right]^{p_{mn}}\right.\right.\right.$
$\left., z; t) < \infty\}\right)$

Proof: Let $x = (x_{mn}) \in W(\tau, \Lambda^2)_{\nabla}$. Then there exists a constant $T_1 > 0$ such

that
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right\}\right)\leq$
$sup\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right\}\right)\leq T_1$ for all $r,s\in\mathbb{N}$. Therefore we have
$x=(x_{mn})\in\Lambda^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$. Conversely, let $x=(x_{mn})\in\Lambda^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$. Then there exists a constant $T_2>0$ such that
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right\}\right)\leq T_2$ for all m,n and r,s. So,
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right\}\right)\leq$
$T_2\,\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}1\leq T_2$, for all m,n and r,s. Thus $x=(x_{mn})\in W\left(\tau,\Lambda^2\right)_\nabla$

4.7. Theorem. $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ be a almost $(\lambda_m\mu_n)$ Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) in $(X,F,*)$ is $\nabla-$ statistically convergent if and only if it is $\nabla-$statistically Cauchy

Proof: Let $x=(x_{mn})$ be a $\nabla-$statistically convergent sequence in
$\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$. Let $\epsilon>0$ be given. Choose $s>0$ such that

$$(1-s)*(1-s)>1-\epsilon \tag{4.1}$$

is satisfied.
For $t>0$ and non-zero $z\in\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ define
$A\left(s,t\right)=\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}}\right.\right.\right.$
$\left.\left.\left.,z;\frac{t}{2}\right)\leq 1-s\right\}\right)$ and
$A^c\left(s,t\right)=\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}}\right.\right.\right.$
$\left.\left.\left.,z;\frac{t}{2}\right)>1-s\right\}\right)$.
It follows that $\delta_\nabla\left(A\left(s,t\right)\right)=0$ and consequently $\delta_\nabla\left(A^c\left(s,t\right)\right)=1$. Let $\eta\in A^c\left(s,t\right)$. Then

$$F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;\frac{t}{2}\right)\leq 1-s \tag{4.2}$$

$B\left(\epsilon,t\right)=\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\right.\right.$

$\leq 1-\epsilon\})$.
It is enough to prove that $B(\epsilon,t)\subseteq A(s,t)$. Let $a,b\in B(\epsilon,t)$, then for non-zero $z\in\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$.

$$\frac{1}{\lambda_m\mu_n}\sum_{a\in I_{r,s}}\sum_{b\in I_{r,s}}F\left(\left[M\left(x_{a,b}-x_{c,d}\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\leq 1-\epsilon. \tag{4.3}$$

If

$$\tfrac{1}{\lambda_m\mu_n}\sum_{a\in I_{r,s}}\sum_{b\in I_{r,s}}F\left(\left[M\left(x_{a,b}-x_{c,d}\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\leq 1-\epsilon.$$

then we have

$$\tfrac{1}{\lambda_m\mu_n}\sum_{a\in I_{r,s}}\sum_{b\in I_{r,s}}F\left(\left[M\left(x_{a,b}-0\right)^{1/m+n}\right]^{p_{mn}},z;\tfrac{t}{2}\right)\leq 1-s$$

and therefore $a,b\in A(s,t)$. As otherwise that is if
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{a\in I_{r,s}}\sum_{b\in I_{r,s}}F\left(\left[M\left((a+b)!\left|x_{a,b}\right|-0\right)^{1/a+b}\right]^{p_{ab}},z;\frac{t}{2}\right)>1-s\right\}\right)$
then by (4.1),(4.2) and (4.3) we get

$$\begin{aligned}&1-\epsilon\geq\tfrac{1}{\lambda_m\mu_n}\sum_{a\in I_{r,s}}\sum_{b\in I_{r,s}}F\left(\left[M\left(x_{a,b}-x_{c,d}\right)^{1/m+n}\right]^{p_{mn}},z;t\right)\\&\geq\left(\tfrac{1}{\lambda_m\mu_n}\left\{\sum_{a\in I_{r,s}}\sum_{b\in I_{r,s}}F\left(\left[M\left((a+b)!\left|x_{a,b}\right|-0\right)^{1/a+b}\right]^{p_{ab}},z;\tfrac{t}{2}\right)>1-s\right\}\right)*\\&\left(\tfrac{1}{\lambda_m\mu_n}\left\{\sum_{c\in I_{r,s}}\sum_{c\in I_{r,s}}F\left(\left[M\left((c+d)!\left|x_{c,d}\right|-0\right)^{1/c+d}\right]^{p_{cd}},z;\tfrac{t}{2}\right)>1-s\right\}\right)\\&\geq(1-s)*(1-s)>1-\epsilon\end{aligned}$$

which is not possible. Thus $B(\epsilon,t)\subset A(s,t)$. Since $\delta_\nabla(A(s,t))=0$, it follows that $\delta_\nabla(B(\epsilon,t))=0$. This shows that (x_{mn}) is $\nabla-$statistically Cauchy.

Conversely, suppose (x_{mn}) is $\nabla-$statistically Cauchy not in $\nabla-$statistically convergent. Then there exists positive integer η and for non-zero
$z\in\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$ such that if we take

$$A(\epsilon,t)=\left(\tfrac{1}{\lambda_m\mu_n}\left\{\sum_{a\in I_{r,s}}\sum_{b\in I_{r,s}}F\left(\left[M\left(x_{a,b}-x_{cd}\right)^{1/a+b}\right]^{p_{ab}},z;t\right)\leq 1-\epsilon\right\}\right)$$

and
$B(\epsilon,t)=\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z;\frac{t}{2}\right)\right.\right.$
$\left.\left.>1-\epsilon\right\}\right)$.
then

$$\delta_\nabla(A(\epsilon,t))=0=\delta_\nabla(B(\epsilon,t))$$

consequently
$$\delta_{\nabla}\left(A^{c}\left(\epsilon,t\right)\right)=1=\delta_{\nabla}\left(B^{c}\left(\epsilon,t\right)\right). \tag{4.4}$$
Since
$\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{a\in I_{r,s}}\sum_{b\in I_{r,s}} F\left(\left[M\left(x_{a,b}-x_{cd}\right)^{1/a+b}\right]^{p_{ab}}, z; t\right)\right\}\right) \geq$
$2\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}}, z; \frac{t}{2}\right)\right\}\right) > 1-\epsilon,$
if
$\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} F\left(\left[M\left((m+n)!\left|x_{m,n}\right|-0\right)^{1/m+n}\right]^{p_{mn}}, z; \frac{t}{2}\right)\right\}\right) > \frac{1-\epsilon}{2}$
then we have
$$\delta_{\nabla}\left(\frac{1}{\lambda_m \mu_n}\left\{\sum_{a\in I_{r,s}}\sum_{b\in I_{r,s}} F\left(\left[M\left(x_{a,b}-x_{cd}\right)^{1/a+b}\right]^{p_{ab}}, z; t\right) > 1-\epsilon\right\}\right)=0$$
that is $\delta_{\nabla}\left(A^{c}\left(\epsilon,t\right)\right)=0$, which contradicts (4.4) as $\delta_{\nabla}\left(A^{c}\left(\epsilon,t\right)\right)=1$. Hence $x=(x_{mn})$ is $\nabla-$statistically convergent.

Competing Interests: The authors declare that there is no conflict of interests regarding the publication of this research paper.

References

[1] B. C. Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)**, (2003), 231-237.

[2] M. Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 523-531.

[3] M. Mursaleen and O. H. H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288(1)**, (2003), 223-231.

[4] M. Mursaleen and O. H. H. Edely,Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 532-540.

[5] N. Subramanian and U. K. Misra, The semi normed space defined by a double gai sequence of modulus function, *Fasciculi Math.*, **46**, (2010).

[6] F. Moricz and B. E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.*, **104**, (1988), 283-294.

[7] A. Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, *Math. Ann.*, **53**, (1900), 289-321.

[8] G. M. Robison, Divergent double sequences and series, *Amer. Math. Soc. Trans.*, **28**, (1926), 50-73.

[9] E. Savas and Richard F. Patterson, On some double almost lacunary sequence spaces defined by Orlicz functions, *Filomat (Niš)*, **19**, (2005), 35-44.

[10] Kuldip Raj and Sunil K. Sharma, Lacunary sequence spaces defined by a Musielak Orlicz function , *Le Matematiche*, **Vol. LXVIII-Fasc. I**, (2013), 33-51.

[11] Francesco Tulone, Regularity of some method of summation for double sequences , *Le Matematiche*, **Vol. LXV-Fasc. II**, (2010), 45-48.

[12] J. Musielak, Orlicz Spaces,Lectures Notes in Math.,1034, *Springer-Verlag*, **1983**.