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On Certain Identities involving q-Series

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Abstract: In this paper, making use of a transformation formula of ${}_2\Psi_2$ due to Bailey [2]. We have established certain Eta function identities, which compliment the works of Berndt and Zhang [3] Bhargava and Somashekara [4] Fine [5].
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1. Introduction, Notations and Definitions:

In this paper we establish certain Eta function identities which compliment the works of Berndt and Zhang [3] Bhargava and Somashekara [4] Fine [5].
We shall use the following known transformation due to Bailey [2]

$$ {}_2\Psi_2\left[\begin{matrix} a, b; q; z \\ c, d \end{matrix}\right] = \frac{[az, d/a, c/b, dq/abz; q]_\infty}{[z, d, q/b, cd/abz; q]_\infty}\, {}_2\Psi_2\left[\begin{matrix} a, abz/d; q; d/a \\ az, c \end{matrix}\right] \tag{1.1} $$

to establish our results involving Eta function.
Taking $z = q/a$ and $b = d/q$ in (6.1.1) we get after some simplification.

$$ \sum_{n=-\infty}^{\infty} \frac{[a, d/q; q]_n (q/a)^n}{[c, d; q]_n} = \frac{[q, d/a, cq/d, q; q]_\infty}{[q/a, d, q^2/d, c; q]_\infty} \tag{1.2} $$

Now replacing d by dq in (1.2), we get

$$ \sum_{n=-\infty}^{\infty} \frac{[a; q]_n (q/a)^n}{[c; q]_n (1 - dq^n)} = \frac{[q, dq/a, c/d, q; q]_\infty}{[q/a, d, q/d, c; q]_\infty} \tag{1.3} $$

As $a \to \infty$ in (1.3), we get

$$ \sum_{n=-\infty}^{\infty} \frac{(-)^n q^{n(n+1)/2} (1 - dq^n)}{[c; q]_n} = \frac{[c/d; q]_\infty [q; q]_\infty^2}{[c, d, q/d; q]_\infty} \tag{1.4} $$

Now, taking c=a in (1.3), we get

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{a^n(1-dq^n)} = \frac{[q, dq/a, a/d, q; q]_\infty}{[a, q/a, d, q/d; q]_\infty} \tag{1.5}$$

Again with c=aq in (1.3) we have

$$\sum_{n=-\infty}^{\infty} \frac{(q/a)^n}{(1-aq^n)(1-dq^n)} = \frac{[q, dq/a, aq/d, q; q]_\infty}{[a, q/a, d, q/d; q]_\infty} \tag{1.6}$$

If we set a=d in (1.6) we get,

$$\sum_{n=-\infty}^{\infty} \frac{(q/a)^n}{(1-aq^n)^2} = \frac{[q; q]_\infty^4}{[a, q/a; q]_\infty^2} \tag{1.7}$$

If we replace q by q^2 and set a=q in (1.7), we get

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{(1-q^{2n+1})^2} = \frac{[q^2; q^2]_\infty^8}{[q; q]_\infty^4} = \frac{\eta^8(2\tau)}{q^{1/2}\eta^4(\tau)} \tag{1.8}$$

Again, replacing q by q^3 in (1.7) and then setting a=q, we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{3n+1})^3} = \frac{[q^3; q^3]_\infty^6}{[q; q]_\infty^2} = \frac{\eta^6(3\tau)}{q^{2/3}\eta^2(\tau)} \tag{1.9}$$

Further, if we replace q by q^5 in (1.7) and take a=q and $a = q^2$, respectively in it. The resulting equations lead to the following interesting relation, after some simplification.

$$\left\{\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{(1-q^{5n+1})^2}\right\}\left\{\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1-q^{5n+2})^2}\right\} = \frac{\eta^{10}(5\tau)}{q^2\eta^2(\tau)} \tag{1.10}$$

Next, if we replace q by q^2 and then set $c = q^2$ and d=q in (1.4), we get

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{[q^2; q^2]_n(1-q^{2n+1})} = \frac{\eta^2(2\tau)}{q^{1/8}\eta(\tau)} \tag{1.11}$$

Again, replacing q by q^2 and taking c=q and $d = q^{-1}$ in (1.4), we get

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{[q; q^2]_n(1-q^{2n-1})} = -q^{5/8}\frac{\eta^6(2\tau)}{\eta(\tau)} \tag{1.12}$$

Now, if we replace q by q^3 and then take a=q and $d = q^2$ in (1.6), we get

$$(1-q)\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{3n+1})(1-q^{3n+2})} = \frac{\eta^3(3\tau)}{q^{1/3}\eta(\tau)} \tag{1.13}$$

Next, if we replace q by q^4 and then take $a = q^2$ in (1.7), we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{4n+2})^2} = \frac{[q^4;q^4]_\infty^8}{[q^2;q^2]_\infty^4} = \frac{\eta^8(4\tau)}{q\eta^4\eta(2\tau)} \tag{1.14}$$

which could also be obtained by replacing q by q^2 in (1.8).
Further, replacing q by q^4 and then taking a=q and $d = q^2$ in (1.6),we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1-q^{4n+1})(1-q^{4n+2})} = \frac{\eta^4(4\tau)}{q^{1/2}(1-q)\eta^2(2\tau)} \tag{1.15}$$

Next, if we replace q by q^4 in (1.5) and then put a=q and $d = q^2$ in it, we have

$$\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1-q^{4n+2})} = -\frac{\eta^4(4\tau)}{q^{3/2}\eta^2(2\tau)} \tag{1.16}$$

Again, replacing q by q^4 and setting $a = q^2$ and d=q in (1.5), we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{4n+1})} = \frac{\eta^2(4\tau)}{q^{1/6}\eta^2(2\tau)} \tag{1.17}$$

Now, if we write q^3 for q and set $a = q^2$ and d=q in (1.5), we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{n}}{(1-q^{3n+1})} = \frac{\eta^3(3\tau)}{q^{1/3}\eta(\tau)} \tag{1.18}$$

If we replace c by 0 in (1.4), we get

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{(1-dq^n)} = \frac{[q;q]_\infty^2}{[d,q/d;q]_\infty} \tag{1.19}$$

which is a known result (cf. Andrews and Berndt [1:12.2.9 p. 264]).
Now, if we write q^2 for q in the above result with d replaced by q, we get

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n q^{n(n+1)}}{(1-q^{2n+1})} = \frac{[q^2;q^2]_\infty^4}{[q;q]_\infty^2} = \frac{\eta^4(2\tau)}{q^{1/4}\eta^2(\tau)} \tag{1.20}$$

Again, replacing q by q^3 and then taking $d=q^2$ in (1.19), we get

$$\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{3n(n+1)/2}}{(1-q^{3n+2})}=\frac{[q^3;q^3]_\infty^3}{[q;q]_\infty}=\frac{\eta^3(3\tau)}{q^{1/3}\eta(\tau)} \tag{1.21}$$

Next, replacing q by q^5 and setting d=q in (1.19), we get

$$\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{5n(n+1)/2}}{1-q^{5n+1}}=\frac{[q^5;q^5]_\infty^2}{[q,q^4;q^5]_\infty} \tag{1.22}$$

Also, replacing q by q^5 and taking $d=q^2$ in (1.19), we get

$$\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{5n(n+1)/2}}{1-q^{5n+2}}=\frac{[q^5;q^5]_\infty^2}{[q^2,q^3;q^5]_\infty} \tag{1.23}$$

Now, (1.22) and (1.23) yield the following interesting result involving q-series and continued fraction

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{5n(n+1)/2}}{1-q^{5n+1}}}{\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{5n(n+1)/2}}{1-q^{5n+2}}}=1+\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots \tag{1.24}$$

Also (1.22) and (1.23) lead to

$$\left\{\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{5n(n+1)/2}}{(1-q^{5n+1})}\right\}\left\{\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{5n(n+1)/2}}{(1-q^{5n+2})}\right\}=\frac{[q^5;q^5]_\infty^5}{[q;q]_\infty}=\frac{\eta^5(5\tau)}{q\eta(\tau)} \tag{1.25}$$

Now, q replaced by q^4 and d by q in (1.19) yield

$$\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{2n(n+1)}}{1-q^{4n+1}}=\frac{[q^4;q^4]_\infty^2[q^2;q^2]_\infty}{[q;q]_\infty}=\frac{\eta^2(4\tau)\eta(2\tau)}{q^{3/8}\eta(\tau)} \tag{1.26}$$

Now, q replace by q^4 and d by q^2 in (1.19) lead to

$$\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{2n(n+1)}}{1-q^{4n+2}}=\frac{[q^4;q^4]_\infty^4}{[q^2;q^2]_\infty^2}=\frac{\eta^4(4\tau)}{q^{1/2}\eta^2(2\tau)} \tag{1.27}$$

Finally, if q and c both replaced by q^2 and a and d each by q, we get,

$$\sum_{n=-\infty}^{\infty} \frac{[q;q^2]_n q^n}{[q^2;q^2]_n(1-q^{2n+1})} = \frac{[q^2;q^2]_\infty^4}{[q;q]_\infty^2} = \frac{\eta^4(2\tau)}{q^{1/4}\eta^2(\tau)} \tag{1.28}$$

2. q-Series Transformations:

In this section we shall discuss certain q-series transformations arising out of the results of the section one.

If we take into account the relations (1.9) and (1.18), we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{3n+1})^2} = \left\{ \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{3n+1}} \right\}^2 \tag{2.1}$$

Again, (1.8) and (1.11) lead to

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{(1-q^{2n+1})^2} = \left\{ \sum_{n=-\infty}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{[q^2;q^2]_n(1-q^{2n+1})} \right\}^4 \tag{2.2}$$

Also, (1.14) and (1.15) yield

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{4n+2})^2} = \left\{ (1-q) \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1-q^{4n+1})(1-q^{4n+2})} \right\}^2 \tag{2.3}$$

Further, comparing (1.14) and (1.27), we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{4n+2})^2} = \left\{ \sum_{n=-\infty}^{\infty} \frac{(-)^n q^{2n(n+1)/2}}{(1-q^{4n+2})} \right\}^2 \tag{2.4}$$

It will be wroth while providing the direct proof of the identities established in section 2.

It is evident that we can establish a large number of relations involving Eta functions and q-series leading to q-series identities.

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