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ON CERTAIN POLYNOMIALS ASSOCIATED WITH THE MULTIVARIABLE ALEPH-FUNCTION

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Abstract: In an attempt to unify various bilateral generating functions obtained earlier by Srivastava and Panda [10], Raina [5], and Srivastava and Raina [12], we study here a few new sets of polynomials associated with the multivariable Aleph-function and give certain theorems concerning the generating functions of these polynomials. In the sequel we also show that these theorems can be applied to yield several bilateral generating function for some polynomial sets.

Keywords: Multivariable Aleph-function, classes of multivariable polynomials, multidimensional generating function, multivariable H-function, Aleph-function of two variables, I-function of two variables.

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1. Introduction and Preliminaries

The multivariable Aleph-function is an extension of the multivariable I-function recently defined by C.K. Sharma and Ahmad [7], itself is a generalization of the multivariable H-function defined by Srivastava et al [9,10]. The multivariable Aleph-function is defined by means of the multiple contour integral.
We have,

$$\aleph(z_1,...,z_r)=\aleph^{0,n:m_1,n_1,...,m_r.n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};...;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}$$

$$\left(\begin{array}{c|c} z_1 & [(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,n}],[\tau_i(a_{ji};\alpha_{ji}^{(1)},...,\alpha_{ji}^{(r)})_{n+1,p_i}]:[(c_j^{(1)},\gamma_j^{(1)})_{1,n_1}], \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & ....[\tau_i(b_{ji};\beta_{ji}^{(1)},...,\beta_{ji}^{(r)})_{m+1,q_i}]:[(d_j^{(1)},\delta_j^{(1)})_{1,m_1}], \end{array}\right.$$

$$\left.\begin{array}{c} [\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}})_{n_1+1,p_i^{(1)}}];...;;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}})_{n_r+1,p_i^{(r)}}] \\ \\ [\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}})_{m_1+1,q_i^{(1)}}];...;;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}})_{m_r+1,q_i^{(r)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, ..., s_r) \prod_{k=1}^{r} \phi_k(s_k) z_k^{s_k} ds_1 ... ds_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, ..., s_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k)}{\sum_{i=1}^{R} \left[\tau_i \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k) \right]} \tag{1.2}$$

and

$$\phi_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} \left[\tau_{i^{(k)}} \prod_{l=m_k+1}^{q_i(k)} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{l=n_k+1}^{p_i(k)} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k) \right]} \tag{1.3}$$

where $j = 1$ to r and k=1 to r.
For more details, see Ayant [1,2,3]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as

$$|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi, \text{ where } A_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} -$$

$$\tau_{i^{(k)}} \sum_{j=n_k+1}^{p_i(k)} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{(q_i(k))} \delta_{ji^{(k)}}^{(k)} > 0, \tag{1.4}$$

with $k = 1, ..., r, i = 1, ..., R$ an $i^{(k)} = 1, ..., R^{(k)}$.
The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form,

$$\aleph(z_1, ..., z_r) = 0(|z_1|^{\alpha_1}, ..., |z_r|^{\alpha_r}), max(|z_1|, ..., |z_r|) \to 0$$

$$\aleph(z_1, ..., z_r) = 0(|z_1|^{\beta_1}, ..., |z_r|^{\beta_r}), min(|z_1|, ..., |z_r|) \to \infty$$

where $k = 1, ..., r; \alpha_k = \min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, ..., m_k$ and $\beta_k = \max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, ..., n_k$.
For convenience, we will use the following notations in this paper.

$$V = m_1, n_1; ...; m_r, n_r \tag{1.5}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, ..., p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.6}$$

$$A = \left\{(a_j; \alpha_j^{(1)}, ..., \alpha_j^{(r)})_{1,n}\right\}, \left\{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, ..., \alpha_{ji}^{(r)})_{n+1,p_i}\right\}, \left\{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\right\},$$

$$\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\Big\}, ..., \left\{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\right\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\Big\} \tag{1.7}$$

$$B = \left\{\tau_i(b_{ji}; \beta_{ji}^{(1)}, ..., \beta_{ji}^{(r)})_{m+1,q_i}\right\}, \left\{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\right\},$$

$$\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\Big\}, ..., \left\{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\right\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\Big\} \tag{1.8}$$

2. Generating functions for a general class of polynomials

Let $f[(z_s)]$ be a function of several complex variables $z_1, ..., z_s$ defined formally by the power series

$$f[(z_s)] = \sum_{k_1,...,k_s=0}^{\infty} C[(k_s)] \prod_{i=1}^{s} \frac{z_i^{k_i}}{k_i!} \tag{2.1}$$

where the coefficients $C[(k_i)], k_i \geq 0, i = 1, ..., s$ are arbitrary constants real or complex.
Also let a class of polynomials $Q_{(n_s);(q_s)}^{[(\alpha_s);(\beta_s)]}[(\lambda_s); (x_s); (y_r)]$ be defined by

$$Q_{(n_s);(q_s)}^{[(\alpha_s);(\beta_s)]}[(\lambda_s); (x_s); (y_r)] = \sum_{k_1=0}^{[n_1/q_1]} ... \sum_{k_s=0}^{[n_s/q_s]} C[(k_s)] \prod_{i=1}^{s} \left\{(-n_i)_{q_i k_i} \frac{x_i^{k_i}}{k_i!}\right\}$$

$$\aleph_{p_i+s,q_i+s,\tau_i;R:W}^{0,n+s:V}\left(\begin{array}{c|c} y_1 & (-\alpha_i - (\beta_i+1)n_i - \lambda_i k_i : \sigma_i', ..., \sigma_i^{(r)})_{1,s}, A \\ \cdot & \cdots \\ \cdot & \\ \cdot & \\ y_r & (-\alpha_i - \beta_i n_i - (\lambda_i + q_i)k_i : \sigma_i', ..., \sigma_i^{(r)})_{1,s}, B \end{array}\right) \tag{2.2}$$

Then our first multidimensional generating function is given by

Theorem 1. With the function $[f[(z_s)]]$ defined by (2.1), let

$$\theta[(n_s), (q_s); (\alpha_s), (\beta_s), (\gamma_s); (\lambda_s), (x_s), (y_r)]$$

$$= \sum_{k_1,...,k_s=0}^{\infty} C[(k_s)] \prod_{i=1}^{s} \left\{ \frac{\gamma_i}{\gamma_i + (\beta_i+1)_{q_i k_i}} \frac{x_i^{(k_i)}}{k_i!} \begin{pmatrix} n_i + q_i k_i + \gamma_i/(\beta_i+1) \\ n_i \end{pmatrix}^{-1} \right\}$$

$$\aleph^{0,n+s:V}_{p_i+s,q_i+s,\tau_i;R:W} \left(\begin{array}{c|c} y_1 & (\gamma_i - \alpha_i - \lambda_i k_i : \sigma'_i, ..., \sigma_i^{(r)})_{1,s}, A \\ \cdot & \\ \cdot & \cdots \\ \cdot & \\ y_r & (\gamma_i - \alpha_i + n_i - \lambda_i k_i : \sigma'_i, ..., \sigma_i^{(r)})_{1,s}, B \end{array} \right) \tag{2.3}$$

and also let

$$\phi[(x_s),(y_r);(v_s)] = \sum_{n_1,...,n_s=0}^{\infty} \theta[(n_s),(q_s);(\alpha_s),(\beta_s),(\gamma_s);(\lambda_s),(x_s),(y_r)] \prod_{i=1}^{s} \frac{v_i^{n_i}}{n_i!} \tag{2.4}$$

Then

$$\sum_{n_1,...,n_s=0}^{\infty} \prod_{i=1}^{s} \left\{ \frac{\gamma_i}{\gamma_i + (\beta_i+1)_{n_i}} \frac{x_i^{k_i}}{k_i!} \right\} Q^{[(\alpha_s);(\beta_s)]}_{(n_s);(q_s)}[(\lambda_s);(x_s);(y_r)] \prod_{i=1}^{s} \frac{t_i^{n_i}}{n_i!}$$

$$= \prod_{i=1}^{s} (1+v_i)^{\alpha_i} \phi[x_1(-v_1)^{q_1}(1+v_1)^{\lambda_1}, ..., x_s(-v_s)^{q_s}(1+v_s)^{\lambda_s},$$

$$y_1 \prod_{i=1}^{s}(1+v_i)^{\sigma'_i}, ..., y_r \prod_{i=1}^{s}(1+v_i)^{\sigma_i^{(r)}}; \frac{-v_1}{1+v_1}, ..., \frac{-v_s}{1+v_s} \Bigg] \tag{2.5}$$

where $v_i = t_i(1+v_i)^{\beta_i+1}, v_i(0) = 0$

Proof. On replacing $Q^{[(\alpha_s);(\beta_s)]}_{(n_s);(q_s)}[(\lambda_s);(x_s);(y_r)]$ by its equivalent series from the definition (2.2) with the multivariable Aleph-function expressed in terms of its Mellin-Barnes contour integral representation (1.1) in the left member of (2.5), changing the order of summation and integration which is assumed to be permissible and then applying a result due to Srivastava and Panda [11, theorem 3, page 34], and finally interpreting the resulting expression by means of (1.1), we arrive at the desired result of (2.5).

3. Two more classes of general polynomials

The polynomials $T^{[(\alpha_s);(\beta_s)]}_{(n_s);(q_s)}[(\lambda_s);(x_s);(y_r)]$: Associated with the power series (2.1), we may define a new set of general polynomials in the form

$$T^{[(\alpha_s);(\beta_s)]}_{(n_s);(q_s)}[(\lambda_s);(x_s);(y_r)] = \sum_{k_1=0}^{[n_1/q_1]} ... \sum_{k_s=0}^{[n_s/q_s]} C[(k_s)] \prod_{i=1}^{s} \{(-n_i)_{q_i k_i} x_i^{k_i}\}$$

$$\aleph^{0,n+s:V}_{p_i+s,q_i+s,\tau_i;R:W}\left(\begin{array}{c|c} y_1 & (1-\alpha_i-(\beta_i+1)n_i-\lambda_i k_i:\sigma'_i,...,\sigma^{(r)}_i)_{1,s},A \\ . & \cdots \\ . & \\ . & \\ y_r & (-\alpha_i-\beta_i n_i-(\lambda_i+q_i)k_i:\sigma'_i,...,\sigma^{(r)}_i)_{1,s},B \end{array}\right) \quad (3.1)$$

the following theorem gives a generating function of type (2.5) $\gamma_i=\alpha_i, i=1,...,s$ with for this set of polynomials.

Theorem 2. If we let

$$\zeta[(x_s),(y_r)]=\sum_{k_1,...,k_s=0}^{\infty}C[(k_s)]\prod_{i=1}^{s}z_i^{k_i}$$

$$\aleph^{0,n+s:V}_{p_i+s,q_i+s,\tau_i;R:W}\left(\begin{array}{c|c} y_1 & (1-\alpha_i-(\beta_i+1)q_i k_i-\lambda_i k_i:\sigma'_i,...,\sigma^{(r)}_i)_{1,s},A \\ . & \cdots \\ . & \\ . & \\ y_r & (-\alpha_i-(\beta_i+1)q_i k_i-\lambda_i k_i:\sigma'_i,...,\sigma^{(r)}_i)_{1,s},B \end{array}\right) \quad (3.2)$$

then

$$\sum_{n_1,...,n_s=0}^{\infty}T^{[(\alpha_s);(\beta_s)]}_{(n_s);(q_s)}[(\lambda_s);(x_s);(y_r)]\prod_{i=1}^{s}\frac{t_i^{n_i}}{n_i!}=\prod_{i=1}^{s}(1+v_i)^{\alpha_i}$$

$$\zeta\left[x_1(-v_1)^{q_1}(1+v_1)^{\lambda_1},...,x_s(-v_s)^{q_s}(1+v_s)^{\lambda_s},y_1\prod_{i=1}^{s}(1+v_i)^{\sigma'_i},...,y_r\prod_{i=1}^{s}(1+v_i)^{\sigma^{(r)}_i}\right] \quad (3.3)$$

The proof of theorem 2 use the similar method that theorem 1.
The polynomials $\Lambda^{[(\alpha_s);(\beta_s)]}_{(n_s);(q_s)}[(\lambda_s);(x_s);(y_r)]$: Associated with the power series (2.1), we may define a new set of general polynomials in the form

$$\Lambda^{[(\alpha_s);(\beta_s)]}_{(n_s);(q_s)}[(\lambda_s);(x_s);(y_r)]=\sum_{k_1,...,k_s=0}^{\infty}C[(k_s)]\prod_{i=1}^{s}\left\{\frac{1}{(n_i+1)_{\mu_i k_i}}\frac{x_i^{k_i}}{k_i!}\right\}$$

$$\aleph^{0,n+s:V}_{p_i+s,q_i+s,\tau_i;R:W}\left(\begin{array}{c|c} y_1 & (-\alpha_i-(\beta_i+1)n_i-\lambda_i k_i:\sigma'_i,...,\sigma^{(r)}_i)_{1,s},A \\ . & \cdots \\ . & \\ . & \\ y_r & (-\alpha_i-\beta_i n_i-(\lambda_i-\mu_i)k_i:\sigma'_i,...,\sigma^{(r)}_i)_{1,s},B \end{array}\right) \quad (3.4)$$

and

$$\Lambda^{[(\alpha_s);(\beta_s)]}_{(-n_s);(q_s)}[(\lambda_s);(x_s);(y_r)] = \sum_{k_1\geq[n_1/\mu_1]}^{\infty} \cdots \sum_{k_s\geq[n_s/\mu_s]}^{\infty} C[(k_s)] \prod_{i=1}^{s}\left\{\frac{1}{(-n_i+1)_{\mu_i k_i}}\frac{x_i^{k_i}}{k_i!}\right\}$$

$$\aleph^{0,n+s:V}_{p_i+s,q_i+s,\tau_i;R:W}\left(\begin{array}{c|c} y_1 & (-\alpha_i-(\beta_i+1)n_i-\lambda_i k_i:\sigma_i',...,\sigma_i^{(r)})_{1,s}, A \\ \cdot & \cdots \\ \cdot & \\ \cdot & \\ y_r & (-\alpha_i+\beta_i n_i-(\lambda_i-\mu_i)k_i:\sigma_i',...,\sigma_i^{(r)})_{1,s}, B \end{array}\right) \tag{3.5}$$

By using the known result [11, theorem, page 38], we can establish the following theorem giving a Laurent series expansion involving these polynomials.

Theorem 3. In terms of the coefficients $C[(k_s)], k_i \geq 0, i = 1,...,s$, given by (2.1), let θ and ϕ be defined by equations (2.3) and (2.4), respectively, with $q_i = -\mu_i, i = 1,...,s$. Then the polynomials defined by (3.4) and (3.5) are generated by

$$\sum_{n_1,...,n_s=-\infty}^{\infty}\prod_{i=1}^{s}\left\{\frac{\gamma_i}{\gamma_i+(\beta_i+1)_{n_i}}\frac{t_i^{n_i}}{n_i!}\right\}\Lambda^{[(\alpha_s);(\beta_s)]}_{(n_s);(q_s)}[(\lambda_s);(x_s);(y_r)] = \prod_{i=1}^{s}(1+v_i)^{\alpha_i}$$

$$\phi\left[x_1(v_1)^{-\mu_1}(1+v_1)^{\lambda_1},...,x_s(v_s)^{-\mu_s}(1+v_s)^{\lambda_s}, y_1\prod_{i=1}^{s}(1+v_i)^{\sigma_i'},...,y_r\prod_{i=1}^{s}(1+v_i)^{\sigma_i^{(r)}}\right.$$

$$\left.;\frac{-v_1}{1+v_1},...,\frac{-v_s}{1+v_s}\right] \tag{3.6}$$

Remarks. We obtain the same relations with the multivariable H-function defined by Srivastava and Panda [9,10], see Munot and Mathur [4] for more details.

If $r = 2$, the multivariable Aleph-function reduces to Aleph-function of two variables defined by Sharma [6], and we obtain the same relations.

If $r = 2$ and $\tau_i, \tau_{i'}, \tau_{i''} \to 1$, the multivariable Aleph-function reduces to I-function of two variables defined by Sharma and Mishra [8] and we have the similar formulae.

4. Conclusion

Specializing the parameters of the multivariable Aleph-function, we can obtain a large number of generating functions for a general class of polynomials involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

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