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ON A SPECIAL WEAKLY PROJECTIVELY SYMMETRIC RIEMANNIAN MANIFOLD

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Abstract: The notion of a weakly symmetric and weakly projective symmetric Riemannian manifolds have been introduced and studied by L. Tamassy and T. Q. Binh ([7], [8]). Recently, Singh and Khan [5] introduced the notion of special weakly symmetric Riemannian manifolds and denoted such manifold by $(SWS)_n$. In this paper, I have studied the nature of Ricci tensor R of type $(1,1)$ in a special weakly projective symmetric Riemannian manifold $(SWPS)_n$ and have investigated some interesting result on $(SWPS)_n$.

Keywords and Phrases: Projective curvature tensor, Ricci tensor, Einstein manifold, Special weakly projective symmetric Riemannian manifold.

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1. Introduction

Let M^n be an n-dimensional Riemannian manifold and $\chi(M)$ denote the set of differentiable vector fields on M^n. Let $K(X,Y,Z)$ be the Riemannian curvature tensor of type $(1,3)$ for $X,Y,Z \in \chi(M)$. A non-flat Riemannian manifold (M^n,g), $(n \geq 2)$ is called a special weakly symmetric Riemannian manifold [5], if the curvature tensor K of type $(1,3)$ satisfies the condition

$$(D_XK)(Y,Z,V) = 2\alpha(X)K(Y,Z,V) + \alpha(Y)K(X,Z,V) + \alpha(Z)K(Y,X,V) + \alpha(V)K(Y,Z,X), \tag{1.1}$$

where α is a non-zero 1- form. ρ is associated vector field such that

$$\alpha(X) = g(X,\rho), \tag{1.2}$$

for every vector field X and D denotes the operator of covariant differentiation with respect to the metric g. Such a manifold is denoted by $(SWS)_n$. If we replace K by P in (1.1), then it reduces to

$$(D_X P)(Y,Z,V) = 2\alpha(X)P(Y,Z,V) + \alpha(Y)P(X,Z,V) + \alpha(Z)P(Y,X,V) + \alpha(V)P(Y,Z,X), \tag{1.3}$$

where P is the projective curvature tensor defined by (see[5] and [6])

$$P(Y,Z,V) = K(Y,Z,V) - \frac{1}{n-1}[Ric(Z,V)Y - Ric(Y,V)Z]. \tag{1.4}$$

Here Ric is the Ricci tensor of type $(0,2)$. Such an $n-$ dimensional Riemannian manifold shall be called a special weakly projective symmetric Riemannian manifold and such a manifold is denoted by $(SWPS)_n$.

Let

$$'P(X,Y,Z,V) = g(P(X,Y,Z),V), \tag{1.5}$$

then (1.4) reduces to the form

$$'P(X,Y,Z,V) = {}'K(X,Y,Z,V) - \frac{1}{n-1}[Ric(Y,Z)g(X,V) - Ric(X,Z)g(Y,V)], \tag{1.6}$$

where

$$'K(X,Y,Z,V) = g(K(X,Y,Z),V). \tag{1.7}$$

Let

$$h(X,V) = {}'P(X,e_i,e_i,V), \tag{1.8}$$

then (1.6) gives

$$h(X,V) = \frac{n}{n-1}Ric(X,V) - \frac{r}{n-1}g(X,V), \tag{1.9}$$

where r is the scalar curvature.

If a Riemannian manifold is an Einstein manifold, then

$$Ric(X,Y) = \lambda g(X,Y), \tag{1.10}$$

where λ is constant. From(1.10), we have

$$R(X) = \lambda X, \tag{1.11}$$

where R is the Ricci tensor of type $(1,1)$ and is defined by

$$g(R(X),Y) = Ric(X,Y). \tag{1.12}$$

Contracting (1.11), we get

$$r = n\lambda \tag{1.13}$$

The above results will be used in the next section.

2. Existence of a $(SWPS)_n$

Let (M^n, g) be a $(SWPS)_n$. Taking covariant derivative of (1.4) with respect to X and then using (1.3), we get

$$2\alpha(X)P(Y,Z,V) + \alpha(Y)P(X,Z,V) + \alpha(Z)P(Y,X,V) + \alpha(V)P(Y,Z,X)$$
$$= (D_XK)(Y,Z,V) - \frac{1}{n-1}[(D_XRic)(Z,V)Y - (D_XRic)(Y,V)Z]. \tag{2.1}$$

By virtue of (1.4), the equation (2.1) reduces to

$$(D_XK)(Y,Z,V) - 2\alpha(X)K(Y,Z,V) - \alpha(Y)K(X,Z,V) - \alpha(Z)K(Y,X,V)$$
$$-\alpha(V)K(Y,Z,X) - \frac{1}{n-1}[(D_XRic)(Z,V)Y - (D_XRic)(Y,V)Z$$
$$-2\alpha(X)\{Ric(Z,V)Y - Ric(Y,V)Z\} - \alpha(Y)\{Ric(Z,V)X - Ric(X,V)Z\}$$
$$-\alpha(Z)\{Ric(X,V)Y - Ric(Y,V)X\} - \alpha(V)\{Ric(Z,X)V - Ric(Y,X)Z\}] = 0 \tag{2.2}$$

Permuting equation (2.2) twice with respect to X, Y, Z; adding the three obtained equations and using Bianchi's first and second identities; symmetric property of Ricci tensor and the skew-symmetric properties of curvature tensor, we get

$$(D_XRic)(Z,V)Y + (D_YRic)(X,V)Z + (D_ZRic)(Y,V)X - (D_XRic)(Y,V)Z$$
$$-(D_YRic)(Z,V)X - (D_ZRic)(X,V)Y = 0. \tag{2.3}$$

Contracting (2.3) with respect to X, we get

$$(D_ZRic)(Y,V) - (D_YRic)(Z,V) = 0. \tag{2.4}$$

Consequently relation (2.4) gives

$$(D_ZR)(Y) - (D_YR)(Z) = 0. \tag{2.5}$$

This leads us to the following:

Theorem 1. *The Ricci tensor of type* $(1, 1)$ *is closed in special weakly projectively symmetric Riemannian manifold.*

Contracting (2.5) with respect to Y*, we get*

$$Zr = 0,$$

which shows that the scalar curvature r *is constant.* Thus we have the following result:

Theorem 2. *The scalar curvature* r *is constant in case of a special weakly projectively symmetric Riemannian manifold.*

Now, let a non-flat Riemannian manifold (M^n, g) *be a* $(SWPS)_n$ *and let it admit a unit parallel vector field* V*, that is*

$$D_X V = 0 \tag{2.6}$$

Applying Ricci identity to (2.6), we get

$$K(X, Y, V) = 0 \tag{2.7}$$

which in view of (1.7) gives

$$'K(X, Y, Z, V) = 0, \tag{2.8}$$

and therefore

$$Ric(X, V) = 0. \tag{2.9}$$

By virtue of (2.8) and (2.9), the relation (1.6) reduces to

$$'P(X, Y, Z, V) = 0. \tag{2.10}$$

Using (1.8) in (2.10), we get

$$h(X, V) = 0. \tag{2.11}$$

Taking an account of (2.11) and the fact that V *is a unit parallel vector field, it follows from (2.5) that*

$$r = 0 \tag{2.12}$$

Now from (1.8) and (1.3), we have

$$(D_Z h)(X, V) = (D_Z\ 'P)(X, e_i, e_i, V)$$

$$= 2\alpha(Z)\ 'P(X, e_i, e_i, V) + \alpha(X)\ 'P(Z, e_i, e_i, V) + \alpha(e_i)\ 'P(X, Z, e_i, V)$$

$$+\alpha(e_i)\ 'P(X,e_i,Z,V)+\alpha(V)\ 'P(X,e_i,e_i,Z). \tag{2.13}$$

Using (1.6), (2.6), (2.9), (2.11) and (2.12), the relation (2.13) takes the form

$$\alpha(V)Ric(X,Z)=0. \tag{2.14}$$

Since $\alpha(V)\neq 0$, it follows from (2.14) that

$$Ric(X,Z)=0. \tag{2.15}$$

By virtue of the equation (2.15), the equation (1.4) gives

$$P(X,Y,Z)=K(X,Y,Z). \tag{2.16}$$

But by virtue of (1.3) and (2.16), the relation (1.1) holds, that is, a special weakly projective symmetric Riemannian manifold $(SWPS)_n$ reduces to a $(SWS)_n$). Thus, we have the following result:

Theorem 3. *If a $(SWPS)_n$ admits a unit parallel vector field, then it is a $(SWS)_n$.*

By virtue of (1.10), the equation (1.4) reduces to the form

$$P(Y,Z,V)=K(Y,Z,V)-\frac{\lambda}{n-1}[g(Z,V)Y-g(Y,V)Z]. \tag{2.17}$$

Taking covariant derivative of (2.17) with respect to X, we get

$$(D_XP)(Y,Z,V)=(D_XK)(Y,Z,V). \tag{2.18}$$

Using (1.3) in (2.18), we get

$$(D_XK)(Y,Z,V)=2\alpha(X)P(Y,Z,V)+\alpha(Y)P(X,Z,V)$$
$$+\alpha(Z)P(Y,X,V)+\alpha(V)P(Y,Z,X) \tag{2.19}$$

By virtue of (2.17), the equation (2.19) reduces to the from

$$(D_XK)(Y,Z,V)=2\alpha(X)\left[K(Y,Z,V)-\frac{\lambda}{n-1}\{g(Z,V)Y-g(Y,V)Z\}\right]$$
$$+\alpha(Y)\left[K(X,Z,V)-\frac{\lambda}{n-1}\{g(Z,V)X-g(X,V)Z\}\right]$$
$$+\alpha(Z)\left[K(Y,X,V)-\frac{\lambda}{n-1}\{g(X,V)Y-g(Y,V)X\}\right]$$

$$+\alpha(V)\left[K(Y,Z,X)-\frac{\lambda}{n-1}\{g(Z,X)Y-g(Y,X)Z\}\right]$$

From the above we can state the following:

Theorem 4. *The necessary and sufficient condition for an Einstein $(SWPS)_n$ to be a $(SWS)_n$ is that*

$$[2\alpha(X)Y+\alpha(Y)X]g(Z,V)-[2\alpha(X)Z+\alpha(Z)X]g(Y,V)$$

$$+[\alpha(Z)Y-\alpha(Y)Z]g(X,V)+\alpha(V)[g(Z,X)Y-g(Y,X)Z]=0.$$

3. Manifold satisfying $P(Y,Z,V)=0$

Let (M^n,g) be a projectively flat, that is, $P(Y,Z,V)=0$, then the relation (1.4) reduces to

$$K(Y,Z,V)=\frac{1}{n-1}[Ric(Z,V)Y-Ric(Y,V)Z]. \tag{3.1}$$

Taking covariant derivative of (3.1) with respect to X, we have

$$(D_XK)(Y,Z,V)=\frac{1}{n-1}[(D_XRic)(Z,V)Y-(D_XRic)(Y,V)Z]. \tag{3.2}$$

Permuting equation (3.2) twice with respect to X,Y,Z; adding the three obtained equations and then using Bianchi's second identity, we have

$$(D_XRic)(Z,V)Y+(D_YRic)(X,V)Z+(D_ZRic)(Y,V)X$$

$$-(D_XRic)(Y,V)Z-(D_YRic)(Z,V)X-(D_ZRic)(X,V)Y=0. \tag{3.3}$$

An $n-$dimensional Riemannian manifold is called a special weakly Ricci symmetric manifold (see [3]), if the Ricci tensor Ric of type $(0,2)$ satisfies the condition

$$(D_XRic)(Y,V)=2\alpha(X)Ric(Y,Z)+\alpha(Y)Ric(X,Z)+\alpha(Z)Ric(Y,X), \tag{3.4}$$

where α is a non-zero 1-form. Such a manifold is denoted by $(SWRS)_n$. Now, using (3.4) in (3.3), we have

$$\alpha(X)Ric(Z,V)Y+\alpha(Y)Ric(X,V)Z+\alpha(Z)Ric(Y,V)X$$

$$-\alpha(X)Ric(Y,V)Z-\alpha(Y)Ric(Z,V)X-\alpha(Z)Ric(X,V)Y=0. \tag{3.5}$$

Contracting (3.5) with respect to X, we have

$$\alpha(Z)Ric(Y,V)-\alpha(Y)Ric(Z,V)=0. \tag{3.6}$$

Consequently (3.6) gives

$$\alpha(Z)R(Y) - \alpha(Y)R(Z) = 0.$$

Hence, we can state the following:

Theorem 5. *In a projectively flat $(SWRS)_n$, the 1-from α is collinear with the Ricci tensor R.*

Taking $Y = V = e_i$ in (3.6) and performing a summation over i, we get

$$\sum_{i=1}^{n}[\alpha(Z)Ric(e_i, e_i) - \alpha(e_i)Ric(Z, e_i)] = 0$$

or

$$nc\alpha(Z) - \alpha(e_i)c\langle e_i, Z\rangle = 0 \quad or \quad c[n\alpha(Z) - \alpha(Z)] = 0.$$

By virtue of $c \neq 0$, the above relation reduces to $(n-1)\alpha(Z) = 0$. Thus, this leads us to the following:

Theorem 6: *If a projectively flat Riemannian manifold admits a $(SWRS)_n$, then the $1-$ form α must vanish.*

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