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SOME FIXED POINT RESULTS IN CONE METRIC SPACES FOR RATIONAL CONTRACTIONS

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Abstract: In this paper, we prove the some fixed point theorem in cone metric spaces for rational expression in normal cone setting. Our results generalized the main result of Dass, Gupta[11].

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1. Introduction

The Banach contraction principle with rational expressions have been expanded and some fixed and common fixed point theorems have been obtained in [1],[2]. Huang and Zhang [3] initiated cone metric space, which is a generalized of metric spaces, by substituting the real number with ordered Banach spaces. They have considered convergence in cone metric spaces, introduction completeness of cone metric spaces, and proved a Banach contraction mapping theorems, and some other fixed point theorems involving contractive type mapping in cone metric spaces using normality condition. Later, various authors have proved some common fixed point theorems with normal and non-normal cone in these spaces [4], [5], [6], [7], [8]. Quite recently Muhammad Arshad et al. [9] have introduction almost Jaggi and gupta contraction in Partially ordered metric space to prove the fixed point theorem. In this paper we prove the some fixed point Result in cone metric spaces for rational expression in normal cone setting. Our results generalize the main result of Dass, Gupta [11].

2. Preliminaries

Let E be real Banach space and P be a subset of E. Then P is called a cone if

1. P is closed, nonempty, and satisfies $P \neq \{\theta\}$,
2. $ax + by \in P$ for all $x, y \neq P$ and non-negative real number a, b
3. $x \in P$ and $-x \in P \Rightarrow x = \theta$, i.e., $P \cap (-P) = \theta$, where θ denote the zero element of E and by $intP$ the interior of P.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$. A cone P is a solid cone if $intP \neq \Phi$. We write $\|.\|$ as the norm on E. The cone P is called normal if there is a number $k > 0$ such that $\forall x, y \in E, \theta \leq x \leq y$ implies $\|x\| \leq M\|y\|$. The least positive number k satisfying the above is called the normal constant of P. It is well known that $k \geq 1$.
In the following, we always suppose that E is a Banach space, P is cone in E with int $P \neq \Phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.1 [3]. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ Satisfies:
$(CM1)$ $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
$(CM2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$,
$(CM3)$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.2 [3]. $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset R$, $X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.3 [3]. Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Than

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) Let (X, d) is said to be a complete cone metric space if every Cauchy sequence is convergent X.

Lemma 2.4: Let(X, d) be a cone $b-$metric space and $\{x_n\} \in X$ such that

$$d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$$

Where $0 \leq k < 1$, then sequence $\{x_n\}$ is a Cauchy sequence.

3. Main Results. In 1975, Dass and Gupta [11] proved the following theorem:

Let (X, d) be a complete cone metric space. A self-mapping $T : X \times X \to X$ satisfies the condition

$$d(Tx, Ty) \leq \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) \quad \text{for} \quad x, y \in X.$$

Where $\alpha, \beta \in [0, 1)$ is a constant with $\alpha + \beta < 1$. Then T has a unique fixed point in X. Now we generalize this theorem in cone metric space as.

Theorem 3.1. *Let (X, d) be a complete cone metric space. P a normal cone with normal constant M. A self-mapping $T : X \times X \to X$ satisfies the condition*

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 [d(x, Tx) + d(y, Ty)] \frac{[d(x, y) + d(y, Ty)]}{[d(x, Ty)]}$$

$$+\alpha_3 [d(x, Ty) + d(y, Tx)] \frac{[d(x, y) + d(y, Ty) + d(x, Ty)]^2}{[d(x, Ty)]^2}$$

For all $x, y \in X$ where where $\alpha_1, \alpha_2, \alpha_3 > 0$ with $\alpha_1 + 2\alpha_2 + 8\alpha_3 \leq 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be a arbitrary point. Suppose there is an point $x_1 \in X$ such that $Tx_0 = x_1$. By Induction a sequence $\{x_n\}$ can be defined such that $Tx_n = x_{n+1} (n \geq 0)$. If for some $n, x_n = x_{n+1}$, then x_n is a unique fixed point for mapping T. Therefore there is no need to go further. Suppose $x_{n+1} \neq x_n \forall n \geq 1$, Thus replacing x and y by x_{n-1} and x_n respectively in theorem (3.1), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{[d(x_{n-1}, x_{n+1})]}$$

$$+\alpha_3 [d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})] \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})]^2}{[d(x_{n-1}, x_{n+1})]^2}$$

$$\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]} + \alpha_3[d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})]\frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})]^2}{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]^2}$$

$$\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + 4\alpha_3[d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})]$$

$$(1 - \alpha_2 - 4\alpha_3)d(x_n, x_{n+1}) \leq (\alpha_1 + \alpha_2 + 4\alpha_3)d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq \frac{(\alpha_1 + \alpha_2 + 4\alpha_3)}{(1 - \alpha_2 - 4\alpha_3)}d(x_{n-1}, x_n)$$

$$(\alpha_1 + 2\alpha_2 + 8\alpha_3) \leq 1$$

$$\Rightarrow k = \frac{(\alpha_1 + \alpha_2 + 4\alpha_3)}{(1 - \alpha_2 - 4\alpha_3)} \leq 1, \quad 0 < k < 1$$

and by induction

$$\begin{aligned} d(x_n, x_{n+1}) &\leq kd(x_{n-1}, x_n) \\ &\vdots \\ &\leq k^n d(x_0, x_1) \\ d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+m-1}, x_m) \\ &\leq (k^n + k^{n+1} + k^{n+2} + ... + k^{n+m-1})d(x_0, x_1) \\ &\leq \frac{k^n}{1-k}d(x_0, x_1) \end{aligned}$$

We get $\|d(x_n, x_m)\| \leq M\frac{k^n}{1-k}\|d(x_0, x_1)\|$ which implies that $d(x_n, x_m) \to 0$ as $n \to \infty$ Hence x_n is a Cauchy sequence, so by completeness of X this sequence must be convergent in X.
Therefore sequence $\{x_n\}$ converses to a point $x^* \in X$. Now we prove that x^* is the unique fixed point of T.

$$d(x^*, Tx^*) \leq [d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \leq [d(x^*, x_{n+1}) + d(Tx_n, Tx^*)]$$

$$\leq d(x^*, Tx_n) + d(Tx_n, Tx^*)$$

$$\leq d(x^*, Tx_n) + \alpha_1 d(x_n, x^*) + \alpha_2[d(x_n, Tx_n) + d(x^*, Tx^*)]\frac{[d(x_n, x^*) + d(x^*, Tx^*)]}{[d(x_n, x^*) + d(x^*, Tx^*)]}$$

$$+ \alpha_3[d(x_n, Tx^*) + d(x^*, Tx^*)]\frac{[d(x_n, x^*) + d(x^*, Tx^*) + d(x_n, x^*) + d(x^*, Tx^*)]^2}{[d(x_n, x^*) + d(x^*, Tx^*)]^2}$$

$$\leq d(x^*, Tx_n) + \alpha_1 d(x_n, x^*) + \alpha_2[d(x_n, Tx_n) + d(x^*, Tx^*)] + 4\alpha_3[d(x_n, x^*)$$
$$+ d(x^*, Tx^*)]$$
$$\leq d(x^*, x_{n+1}) + \alpha_1 d(x^*, x_n) + \alpha_2[d(x_n, x_{n+1}) + d(x^*, Tx^*)] + 4\alpha_3[d(x_n, Tx^*)$$
$$+ d(x^*, Tx^*)]$$

So using the condition of normality of cone

$$\|d(x^*, Tx^*)\| \leq M(\|d(x^*, x_{n+1})\| + \alpha_1\|d(x_n, x^*)\| + \alpha_2\|d(x_n, x_{n+1})\|$$
$$+\alpha_2\|d(x^*, Tx^*)\| + 4\alpha_3\|d(x_n, x^*)\| + 4\alpha_3\|d(x^*, Tx^*)\|)$$

As $n \to 0$ we have $\|d(x^*, Tx^*)\| \leq 0$. Hence x^* is fixed point of T i.e. $x^* = Tx^*$. Now we will prove that x^* is unique. For that let x' be another fixed point such that $x' = Tx'$. Now from theorem (3.1), we have

$$d(x^*, x') = d(Tx^*, Tx'')$$
$$\leq \alpha_1 d(x^*, x'') + \alpha_2[d(x^*, Tx^{*\prime}) + d(x', Tx'')]\frac{[d(x^*, x'') + d(x', Tx'')]}{[d(x^*, Tx'')]}$$
$$\leq \alpha_3[d(x^*, Tx'') + d(x', Tx^{*\prime})]\frac{[d(x^*, x'') + d(x', Tx'') + d(x^*, Tx'')]}{[d(x^*, Tx'')]}$$
$$d(x^*, x'') \leq \alpha_1 d(x^*, x'') + \alpha_2[d(x^*, x^{*\prime}) + d(x', x'')]\frac{[d(x^*, x'') + d(x', x'')]}{[d(x^*, x'')]}$$
$$+ \alpha_3[d(x^*, x'') + d(x', x^{*\prime})]\frac{[d(x^*, x'') + d(x', x'') + d(x^*, x'')]}{[d(x^*, x'')]}$$
$$\leq \alpha_1 d(x^*, x') + 2\alpha_3 d(x^*, x')$$
$$\leq (\alpha_1 + 2\alpha_3) d(x^*, x')$$
$$\|d(x^*, x')\| \leq M((\alpha_2 + 2\alpha_3)\|d(x^*, x')\|)$$

As $n \to 0$ we have $\|d(x^*, x')\| \leq 0$. Hence $x^* = x'$ is a unique fixed point of T. This completes the proof.

Theorem 3.2. *Let (X,d) be a complete cone metric space. P a normal cone with normal constant M. A self-mapping T:XXX satisfies the condition*

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 \frac{(d(x, Ty))^2}{d(x, y) + d(y, Ty)} + \alpha_3 \frac{(d(y, Ty))^2}{d(x, y) + d(x, Ty)}$$

for all $x, y \in X$ where $\alpha_1\alpha_2, \alpha_3 > 0$ with $\alpha_1 + 2\alpha_2 + 4\alpha_3 \leq 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be a arbitrary point. Suppose there is an point $x_1 \in X$ such that $Tx_0 = x_1$. By Induction a sequence $\{x_n\}$ can be defined such that $Tx_n = x_{n+1}(n \geq 0)$. If for some $n, x_n = x_{n+1}$, then x_n is a unique fixed point for mapping T. Therefore there is no need to go further. Suppose $x_{n+1} \neq x_n \forall n \geq 1$, Thus replacing x and y by x_{n-1} and x_n respectively in theorem (3.2), we have

$$\begin{aligned}
&d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)\\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \frac{(d(x_{n-1}, Tx_n))^2}{d(x_{n-1}, x_n) + d(x_n, Tx_n)} + \alpha_3 \frac{(d(x_n, Tx_n))^2}{d(x_{n-1}, x_n) + d(x_{n-1}, Tx_n)}\\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \frac{(d(x_{n-1}, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})} + \alpha_3 \frac{(d(x_n, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})}\\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \frac{(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}\\
&\qquad + \alpha_3 \frac{(d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})}\\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}\\
&\qquad + \alpha_3 \{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})\}\\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}\\
&\qquad + \alpha_3 \{d(x_{n-1}, x_n) + [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\}\\
&(1 - \alpha_2 - \alpha_3) d(x_n, x_{n+1}) \leq (\alpha_1 + \alpha_2 + 2\alpha_3) d(x_{n-1}, x_n)\\
&d(x_n, x_{n+1}) \leq \frac{(\alpha_1 + \alpha_2 + 2\alpha_3)}{(1 - \alpha_2 - \alpha_3)} d(x_{n-1}, x_n)
\end{aligned}$$

Since $\alpha_1 + 2\alpha_2 + 3\alpha_2 \leq 1 \Rightarrow \frac{(\alpha_1+\alpha_2+3\alpha_3)}{(1-\alpha_2-\alpha_3)} \leq 1 = k, \quad 0 < k < 1$

$$d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n)$$

And by induction

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq k d(x_{n-1}, x_n)\\
&\vdots\\
&\leq k^n d(x_0, x_1)\\
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+m-1}, x_m)\\
&\leq (k^n + k^{n+1} + k^{n+2} + ... + k^{n+m-1}) d(x_0, x_1)\\
&\leq \frac{k^n}{1-k} d(x_0, x_1)
\end{aligned}$$

We get $\|d(x_n, x_m)\| \leq M\frac{k^n}{1-k}\|d(x_0, x_1)\|$ which implies that $d(x_n, x_m) \to 0$ as $n \to \infty$.
Hence x_n is a Cauchy sequence, so by completeness of X this sequence must be convergent in X. Therefore sequence $\{x_n\}$ converses to a point $x^* \in X$. Now we prove that x^* is the unique fixed point of T.

$$\begin{aligned}
&d(x^*, Tx^*) \leq [d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)]\\
&\leq [d(x^*, x_{n+1}) + d(Tx_n, Tx^*)]\\
&\leq d(x^*, Tx_n) + d(Tx_n, Tx^*)\\
&\leq d(x^*, Tx_n) + \alpha_1 d(x^*, x_n) + \alpha_2\frac{[d(x_n, Tx^*)]^2}{d(x_n, x^*) + d(x^*, Tx^*)} + \alpha_3\frac{[d(x^*, Tx^*)]^2}{d(x_n, x^*) + d(x_n, Tx^*)}\\
&\leq d(x^*, Tx_n) + \alpha_1 d(x^*, x_n) + \alpha_2\frac{[d(x_n, x^*) + d(x^*, Tx^*)]^2}{d(x_n, x^*) + d(x^*, Tx^*)} + \alpha_3\frac{[d(x^*, Tx^*)]^2}{d(x^*, Tx^*)}\\
&\leq d(x^*, Tx_n) + \alpha_1 d(x^*, x_n) + \alpha_2[d(x_n, x^*) + d(x^*, Tx^*)] + \alpha_3 d(x^*, Tx^*)\\
&\|d(x^*, Tx^*)\| \leq M(\|d(x^*, Tx_n)\| + \alpha_1\|d(x^*, x_n)\| + \alpha_2\|d(x_n, x^*)\|\\
&\qquad + \alpha_2\|d(x^*, Tx^*)\| + \alpha_3\|d(x^*, Tx^*)\|)
\end{aligned}$$

As $n \to 0$ we have $\|d(x^*, Tx^*)\| \leq 0$. Hence x^* is fixed point of T i.e. $x^* = Tx^*$
Now we will prove that x^* is unique. For that let x' be another fixed point such that $x' = Tx'$. Now from theorem (3.2), we have

$$\begin{aligned}
d(x^*, x') &= d(Tx^*, Tx'')\\
&\leq \alpha_1 d(x^*, x'') + \alpha_2\frac{[d(x^*, Tx'')]^2}{[d(x^*, x'') + d(x', Tx'')]} + \alpha_3\frac{[d(x', Tx'')]^2}{[d(x^*, x'') + d(x^*, Tx'')]}\\
d(x^*, x') &\leq \alpha_1 d(x^*, x'') + \alpha_2\frac{[d(x^*, x'')]^2}{[d(x^*, x'') + d(x', x'')]} + \alpha_3\frac{[d(x', x'')]^2}{[d(x^*, x'') + d(x^*, x'')]}\\
d(x^*, x') &\leq \alpha_1 d(x^*, x'') + \alpha_2 d(x^*, x'')\\
d(x^*, x') &\leq (\alpha_1 + \alpha_2) d(x^*, x'')\\
\|d(x^*, x')\| &\leq (\alpha_1 + \alpha_2)\|d(x^*, x'')\|
\end{aligned}$$

As $n \to 0$ we have $\|d(x^*, x')\| \leq 0$. Hence $x^* = x'$ is a unique fixed point of T.
This completes the proof.

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