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ON CERTAIN PRODUCT FORMULAS FOR BASIC HYPERGEOMETRIC FUNCTIONS

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Abstract: In this paper, making use of certain known summation formulas and modified Bailey transform, we have established product formulas for basic hypergeometric functions.

Keywords and Phrases: Summation formula, transformation formula, product formula, basic hypergeometric function, Bailey transform.

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1. Introduction, Notations and Definitions

The widely-investigated transform, which was discovered by Bailey in 1947, has been used by Bailey [2,3], Slater [7], Andrwes [1], Srivastava, Singh, Singh and Yadav [6], Verma [9], Denis, Singh and Singh [4] and many others to obtain transformations of q-series and also q-series identities. In the present paper, we have used the Bailey transform to obtain product formulas of basic hypergeometric functions.

In this paper, we shall adopt following notations and definitions. The q-rising factorial is defined (for $|q| < 1$) by

$$(a;q)_0 = 1 \qquad \text{and} \qquad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \qquad (n = 1,2,3,\cdots).$$

We also write

$$(a_1, a_2, \cdots, a_r; q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n$$

and

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

A basic (or q-) hypergeometric series is defined by (see [5; p. 347, Eq. 9.4 (272)], see also [8]),

$$\begin{aligned}{}_r\Phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r; & \\ & q; z \\ b_1, b_2, \cdots, b_s; & \end{matrix}\right] \\ = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n z^n}{(q, b_1, b_2, \cdots, b_s; q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r}.\end{aligned}$$

The above q-series converges for all values of z if $r < 1+s$ and for $|z| < 1$, if $r = s+1$.

We now state the Bailey transform as follows (see, for details, [8]; see also [1]):

The Bailey Transform. *If*

$$\beta_n = \sum_{r=0}^{n} \alpha_r \, u_{n-r} \, v_{n+r} \tag{1.1}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r \, v_{r+2n}, \tag{1.2}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \, \gamma_n = \sum_{n=0}^{\infty} \beta_n \, \delta_n, \tag{1.3}$$

where u_r, v_r, α_r and δ_r are arbitrarily chosen sequences of r alone.

If we choose $v_r = 1, \delta_r = z^r$ in (1.1) to (1.3) then Bailey transform takes new form,

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} \tag{1.4}$$

and

$$\gamma_n = z^n \sum_{r=0}^{\infty} u_r z^r \tag{1.5}$$

then under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n z^n, \tag{1.6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ *is arbitrarily chosen sequences of* n *alone.*
We shall make use of following summation formulas

$$ {}_4\Phi_3 \left[\begin{matrix} q^{-n}, -\dfrac{q^{-n}}{xy}, x, y; \\ \\ -xyq, \dfrac{q^{-n}}{x}, \dfrac{q^{-n}}{y} \end{matrix} q; q \right] = \frac{(q;q)_n (xyq;q)_n (x^2q^2;q^2)_m (y^2q^2;q^2)_m}{(xq;q)_n (yq;q)_n (x^2y^2q^2;q^2)_m (q^2;q^2)_m}, \tag{1.7}$$

[10; (2.5), p. 1024]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_2\Phi_1 \left[\begin{matrix} q^{-n}, x; q; -\dfrac{q}{x} \\ \dfrac{q^{-n}}{x} \end{matrix} \right] = \frac{(q;q)_n (x^2q^2;q^2)_m}{(xq;q)_n (q^2;q^2)_m}, \tag{1.8}$$

[10; (2.8), p. 1024]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_2\Phi_1 \left[\begin{matrix} q^{-n}, x; q; -\dfrac{1}{x} \\ \dfrac{q^{-n}}{x} \end{matrix} \right] = \frac{(q;q)_n (x^2q^2;q^2)_m x^{n-2m}}{(xq;q)_n (q^2;q^2)_m}, \tag{1.9}$$

[10; (2.9), p. 1024]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_4\Phi_3 \left[\begin{matrix} q^{-n}, -\dfrac{q^{-n}}{xy}, xq, yq; \\ \\ -xyq, \dfrac{q^{1-n}}{x}, \dfrac{q^{1-n}}{y} \end{matrix} q; q \right] = \frac{(-)^n (q;q)_n (xyq;q)_n (x^2q^2;q^2)_m (y^2q^2;q^2)_m}{q^n (x;q)_n (y;q)_n (x^2y^2q^2;q^2)_m (q^2;q^2)_m}, \tag{1.10}$$

[10; (2.10), p. 1025]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, \dfrac{q^{-n}}{x^2}, y, -y; & \\ & q;q \\ \dfrac{q^{-n}}{x}, -\dfrac{q^{-n}}{x}, y^2q & \end{matrix}\right] = \frac{(q;q)_n(x^2y^2q^2;q^2)_n(x^2q^2;q^2)_m(y^2q^2;q^2)_m}{(x^2q^2;q^2)_n(y^2q;q)_n(x^2y^2q^2;q^2)_m(q^2;q^2)_m}, \tag{1.11} $$

[10; (2.12), p. 1025]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, \dfrac{q^{-n}}{x^2}, 0; q; q \\ \dfrac{q^{-n}}{x}, -\dfrac{q^{-n}}{x} \end{matrix}\right] = \frac{(q;q)_n(x^2q^2;q^2)_m}{(x^2q^2;q^2)_n(q^2;q^2)_m}, \tag{1.12} $$

[10; (2.14), p. 1026]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_2\Phi_2\left[\begin{matrix} q^{-n}, \dfrac{q^{-n}}{x^2}; q; -1 \\ \dfrac{q^{-n}}{x}, -\dfrac{q^{-n}}{x} \end{matrix}\right] = \frac{(q;q)_n(x^2q^2;q^2)_m q^{n(n+1)/2}x^{2n-2m}}{(x^2q^2;q^2)_n(q^2;q^2)_m}, \tag{1.13} $$

[10; (2.15), p. 1026]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, \dfrac{q^{-n}}{x^2}, yq, -yq; & \\ & q;q \\ \dfrac{q^{1-n}}{x}, -\dfrac{q^{1-n}}{x}, y^2q & \end{matrix}\right] = \frac{(-)^n(q;q)_n(x^2y^2q^2;q^2)_n(x^2q^2;q^2)_m(y^2q^2;q^2)_m}{q^n(x^2;q^2)_n(y^2q;q)_n(x^2y^2q^2;q^2)_m(q^2;q^2)_m}. \tag{1.14} $$

[10; (2.16), p. 1026]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, \dfrac{q^{-n}}{x^2}, 0; q; q \\ \dfrac{q^{1-n}}{x}, -\dfrac{q^{1-n}}{x} \end{matrix}\right] = \frac{(-)^n (q;q)_n (x^2q^2;q^2)_m}{q^n (x^2;q^2)_n (q^2;q^2)_m}, \tag{1.15} $$

[10; (2.18), p. 1026]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_2\Phi_2\left[\begin{matrix} q^{-n}, \dfrac{q^{-n}}{x^2}; q; -q^2 \\ \dfrac{q^{1-n}}{x}, -\dfrac{q^{1-n}}{x} \end{matrix}\right] = \frac{(-)^n (q;q)_n (x^2q^2;q^2)_m q^{n(n-1)/2} x^{2n-2m}}{(x^2;q^2)_n (q^2;q^2)_m}, \tag{1.16} $$

[10; (2.19), p. 1026]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, \dfrac{q^{-n}}{x^2}, 0; q; q \\ -\dfrac{q^{-n}}{x}, \dfrac{q^{1-n}}{x} \end{matrix}\right] = \frac{(-)^n (q;q)_n (x^2q^2;q^2)_m x^{n-2m}}{(x;q)_n (-xq;q)_n (q^2;q^2)_m}, \tag{1.17} $$

[10; (2.23), p. 1027]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_2\Phi_2\left[\begin{matrix} q^{-n}, \dfrac{q^{-n}}{x^2}; q; -q \\ -\dfrac{q^{-n}}{x}, \dfrac{q^{1-n}}{x} \end{matrix}\right] = \frac{(-)^n (q;q)_n (x^2q^2;q^2)_m q^{n(n-1)/2} x^{n}}{(x;q)_n (-xq;q)_n (q^2;q^2)_m}, \tag{1.18} $$

[10; (2.24), p. 1027]

where m is the greatest integer $\leq \frac{n}{2}$.

2. Main Results

In this section we establish following product formulas for basic hypergeometric series

(i)

$$
{}_2\Phi_1\left[\begin{array}{l} xq, yq; q; z \\ -xyq \end{array}\right] {}_2\Phi_1\left[\begin{array}{l} x, y; q; -zq \\ -xyq \end{array}\right]
$$
$$
= {}_4\Phi_3\left[\begin{array}{l} xyq, xyq^2, x^2q^2, y^2q^2; q^2; z^2 \\ -xyq, -xyq^2, x^2y^2q^2 \end{array}\right]
$$
$$
+ \frac{z(1-xyq)}{(1+xyq)}\, {}_4\Phi_3\left[\begin{array}{l} xyq^2, xyq^3, xq^2, yq^2; q^2; z^2 \\ -xyq^2, -xyq^3, x^2y^2q^2 \end{array}\right], \tag{2.1}
$$

provided $|z| < 1$.

(ii)

$$
{}_1\Phi_0\left[\begin{array}{l} xq; q; z \\ - \end{array}\right] {}_1\Phi_0\left[\begin{array}{l} x; q; -zq \\ - \end{array}\right]
$$
$$
= {}_1\Phi_0\left[\begin{array}{l} x^2q^2; q^2; z^2 \\ - \end{array}\right] + z\; {}_1\Phi_0\left[\begin{array}{l} x^2q^2; q^2; z^2 \\ - \end{array}\right]
$$
$$
= (1+z)\; {}_1\Phi_0\left[\begin{array}{l} x^2q^2; q^2; z^2 \\ - \end{array}\right], \tag{2.2}
$$

provided $|z| < 1$.

(iii)

$$
{}_1\Phi_0\left[\begin{array}{l} xq; q; z \\ - \end{array}\right] {}_1\Phi_0\left[\begin{array}{l} x; q; -z \\ - \end{array}\right]
$$
$$
= (1+xz)\; {}_1\Phi_0\left[\begin{array}{l} x^2q^2; q^2; z^2 \\ - \end{array}\right]. \tag{2.3}
$$

(iv)

$$
{}_2\Phi_1\left[\begin{array}{l} x, y; q; z \\ -xyq \end{array}\right] {}_2\Phi_1\left[\begin{array}{l} xq, yq; q; -zq \\ -xyq \end{array}\right]
$$
$$
= {}_4\Phi_3\left[\begin{array}{l} xyq, xyq^2, x^2q^2, y^2q^2; q^2; z^2 \\ -xyq, -xyq^2, x^2y^2q^2 \end{array}\right]
$$
$$
- \frac{(1-xyq)z}{(1+xyq)}\, {}_4\Phi_3\left[\begin{array}{l} xyq^2, xyq^3, x^2q^2, y^2q^2; q^2; z^2 \\ -xyq^2, -xyq^3, x^2y^2q^2 \end{array}\right], \tag{2.4}
$$

provided $|z| < 1$.

(v)

$$
\begin{aligned}
&{}_2\Phi_1\left[\begin{matrix} xq, -xq; q; z \\ x^2q \end{matrix}\right] {}_2\Phi_1\left[\begin{matrix} y, -y; q; -zq \\ y^2q \end{matrix}\right] \\
&= {}_4\Phi_3\left[\begin{matrix} xyq, -xyq, xyq^2, -xyq^2; q^2; z^2 \\ x^2q, y^2q, x^2y^2q^2 \end{matrix}\right] \\
&+ \frac{(1-x^2y^2q^2)z}{(1-x^2q)(1-y^2q)}\, {}_4\Phi_3\left[\begin{matrix} xyq^2, -xyq^2, xyq^3, -xyq^3; q^2; z^2 \\ x^2q^3, y^2q^3, x^2y^2q^2 \end{matrix}\right],
\end{aligned} \tag{2.5}
$$

provided $|z| < 1$.

(vi)

$$
\begin{aligned}
&e_q^{-zq}\, {}_2\Phi_1\left[\begin{matrix} xq, -xq; q; z \\ x^2q \end{matrix}\right] \\
&= \sum_{n=0}^{\infty} \frac{z^{2n}}{(q^2, x^2q; q^2)_n} + \frac{z}{(1-x^2q)} \sum_{n=0}^{\infty} \frac{z^{2n}}{(q^2, x^2q^3; q^2)_n},
\end{aligned} \tag{2.6}
$$

where $|z| < 1$.

(vii)

$$
\begin{aligned}
&(z; q)_\infty\, {}_2\Phi_1\left[\begin{matrix} xq, -xq; q; z \\ x^2q \end{matrix}\right] \\
&= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)} x^{2n} z^{2n}}{(q^2, x^2q; q^2)_n} + \frac{xzq}{(1-x^2q)} \sum_{n=0}^{\infty} \frac{q^{n(2n+3)} x^{2n} z^{2n}}{(q^2, x^2q^3; q^2)_n},
\end{aligned} \tag{2.7}
$$

where $|z| < 1$.

(viii)

$$
\begin{aligned}
&{}_2\Phi_1\left[\begin{matrix} x, -x; q; z \\ x^2q \end{matrix}\right] {}_2\Phi_1\left[\begin{matrix} yq, -yq; q; -z/q \\ y^2q \end{matrix}\right] \\
&= {}_4\Phi_3\left[\begin{matrix} xyq, -xyq, xyq^2, -xyq^2; q^2; z^2/q^2 \\ x^2q, y^2q, x^2y^2q^2 \end{matrix}\right] \\
&- \frac{(1-x^2y^2q^2)z}{(1-x^2q)(1-y^2q)q}\, {}_4\Phi_3\left[\begin{matrix} xyq^2, -xyq^2, xyq^3, -xyq^3; q^2; z^2/q^2 \\ x^2q^3, y^2q^3, x^2y^2q^2 \end{matrix}\right],
\end{aligned} \tag{2.8}
$$

provided $|z/q| < 1$.

(ix)

$$e^{-z/q}\ {}_2\Phi_1\left[\begin{array}{l} x,-x;q;z \\ x^2q \end{array}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(z^2/q^2)^n}{(q^2,x^2q;q^2)_n}-\frac{z}{q(1-x^2q)}\sum_{n=0}^{\infty}\frac{(z^2/q^2)^n}{(q^2,x^2q^3;q^2)_n}, \tag{2.9}$$

where $|z/q|<1$.

(x)

$$(z;q)_\infty\ {}_2\Phi_1\left[\begin{array}{l} x,-x;q;z \\ x^2q \end{array}\right]$$
$$=\sum_{n=0}^{\infty}\frac{q^{n(2n-1)}x^{2n}z^{2n}}{(q^2,x^2q;q^2)_n}-\frac{x^2z}{1-x^2q}\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}x^{2n}z^{2n}}{(q^2,x^2q^3;q^2)_n}, \tag{2.10}$$

where $|z|<1$.

(xi)

$$e_q^{-z}\ {}_2\Phi_1\left[\begin{array}{l} x,-xq;q;z \\ x^2q \end{array}\right]$$
$$=\sum_{n=0}^{\infty}\frac{z^{2n}}{(q^2,x^2q;q^2)_n}-\frac{zx}{(1-x^2q)}\sum_{n=0}^{\infty}\frac{z^{2n}}{(q^2,x^2q^3;q^2)_n}, \tag{2.11}$$

where $|z|<1$.

(xii)

$$(z;q)_\infty\ {}_2\Phi_1\left[\begin{array}{l} x,-xq;q;z \\ x^2q \end{array}\right]$$
$$=\sum_{n=0}^{\infty}\frac{q^{n(2n-1)}(xz)^{2n}}{(q^2,x^2q;q^2)_n}-\frac{xz}{1-x^2q}\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}(xz)^{2n}}{(q^2,x^2q^3;q^2)_n}, \tag{2.12}$$

where $|z|<1$.

3. Proof of (2.1)-(2.12)

Putting

$$\left\{u_r=\frac{(xq,yq;q)_r}{(q,-xyq;q)_r},\alpha_r=\frac{(x,y;q)_r(-q)^r}{(q,-xyq;q)_r}\right\},$$

$$\left\{u_r = \frac{(xq;q)_r}{(q;q)_r}, \alpha_r = \frac{(x;q)_r(-q/x)^r}{(q;q)_r}\right\}, \left\{u_r = \frac{(xq;q)_r}{(q;q)_r}, \alpha_r = \frac{(x;q)_r(-1)^r}{(q;q)_r x^r}\right\},$$

$$\left\{u_r = \frac{(x,y;q)_r}{(q,-xyq;q)_r}, \alpha_r = \frac{(xq,yq;q)_r(-1/q)^r}{(q,-xyq;q)_r}\right\},$$

$$\left\{u_r = \frac{(xq,-xq;q)_r}{(q,x^2q;q)_r}, \alpha_r = \frac{(y,-y;q)_r(-q)^r}{(q,y^2q;q)_r}\right\}, \left\{u_r = \frac{(xq,-xq;q)_r}{(q,x^2q;q)_r}, \alpha_r = \frac{(-q)^r}{(q;q)_r}\right\},$$

$$\left\{u_r = \frac{(xq,-xq;q)_r}{(q,x^2q;q)_r}, \alpha_r = \frac{q^{r(r-1)/2}(-1)^r}{(q,y^2q;q)_r q^r}\right\},$$

$$\left\{u_r = \frac{(x,-x;q)_r}{(q,x^2q;q)_r}, \alpha_r = \frac{(yq,-yq;q)_r(-1/q)^r}{(q,y^2q;q)_r}\right\},$$

$$\left\{u_r = \frac{(x,-x;q)_r}{(q,x^2q;q)_r}, \alpha_r = \frac{(-1/q)^r}{(q;q)_r}\right\}, \left\{u_r = \frac{(x,-x;q)_r}{(q,x^2q;q)_r}, \alpha_r = \frac{(-1)^r q^{r(r-1)/2}}{(q;q)_r}\right\},$$

$$\left\{u_r = \frac{(x,-xq;q)_r}{(q,x^2q;q)_r}, \alpha_r = \frac{(-1)^r}{(q;q)_r}\right\},$$

and

$$\left\{u_r = \frac{(x,-xq;q)_r}{(q,x^2q;q)_r}, \alpha_r = \frac{(-1)^r q^{r(r-1)/2}}{(q;q)_r}\right\},$$

one by one in (1.4) and (1.5) and using the summation formulas (1.7), (1.8), (1.9), (1.10), (1.11), (1.12), (1.13), (1.14), (1.15), (1.16), (1.17) and (1.18) respectively we find $\beta_n' s$ and $\gamma_n' s$. Putting these values of $\beta_n' s$ and $\gamma_n' s$ one by one in (1.6) we find (2.1)-(2.12) respectively.

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