J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.1, No.2 (2012), pp. 33-42

Conjugate WP-Bailey pairs and transformation formulae for basic hypergeometric series

Satya Prakash Singh,*Sunil Singh and K.B. Yadav
Department of Mathematics,
T.D.P.G. College, Jaunpur-222002 (U.P.) India.
E-mails: snsp39@yahoo.com, snsp39@gmail.com
*Department of Mathematics & Statistics,
Sydenham College of Commerce and Economics, Mumbai.

(Received October 18, 2012)

Abstract: In this paper we shall make use of conjugate WP-Bailey pairs in order to establish transformation formulae for basic hypergeometric series.
Keywords: Bailey pair, Conjugate Bailey pair, WP-Bailey pair, Conjugate WP-Bailey pair, transformation formula, summation formula.
AMS subject classification code: Primary 33D90, 11A55; Secondary 11F20.

1. Introduction, Notations and Definitions:

For $|q| < 1$ and α real or complex, let

$$[\alpha; q]_n = (1-\alpha)(1-\alpha q)...(1-\alpha q^{n-1}), \qquad n \geq 1$$

and

$$[\alpha; q]_0 = 1.$$

Also,

$$[\alpha; q]_\infty = \prod_{n=0}^{\infty}(1-\alpha q^n)$$

and

$$[\alpha_1, \alpha_2, \alpha_3, ..., \alpha_r; q]_n = [\alpha_1; q]_n[\alpha_2; q]_n[\alpha_3; q]_n...[\alpha_r; q]_n.$$

A basic hypergeometric function is defined as,

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, .., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty}\frac{[a_1, a_2, ..., a_r; q]_n z^n}{[q, b_1, b_2, ..., b_s; q]_n}, \tag{1.1}$$

convergent for $|z| < 1$ and $|q| < 1$.

Transformation formulae for basic hypergeometric functions play fundamental role in q-series identities leading to partition theory and combinatorial identities. The main aim of this paper is to establish transformation formulae for basic hypergeometric series. For this purpose, certain theorems have been establish in section 1 and in section 2, we have defined some WP Bailey pairs.

Bailey in 1949 gave the following very useful lemma for establishing transformation formulae for q-series.

If $\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}$ and $\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}$ then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.2}$$

Choosing $u_r = \frac{1}{[q;q]_r}$ and $v_r = \frac{1}{[aq;q]_r}$ in lemma (1.2) we get,

If

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q;q]_{n-r}[aq;q]_{n+r}} \tag{1.3}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{[q;q]_r [aq;q]_{r+2n}} \tag{1.4}$$

Then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.5}$$

Sequences α_n, β_n satisfying (1.3) are called Bailey pair with respect to the parameter a whereas the sequences γ_n, δ_n satisfying (1.6) are called conjugate Bailey pair.

(i) Choosing $\delta_r = [\rho_1, \rho_2; q]_r \left(\frac{aq}{\rho_1\rho_2}\right)^r$ in (1.4) and calculating γ_n by making use of basic analogue of Gauss summation formula we get the following theorem,

Theorem: If $\beta_n = \frac{1}{[q, aq; q]_n} \sum_{r=0}^{\infty} \frac{(-)^n q^{(1+n)r} [q^{-n};q]_r \alpha_r}{q^{r(r+1)/2} [aq^{1+n};q]_r}$

Then

$$\sum_{r=0}^{\infty} [\rho_1, \rho_2; q]_n \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n$$

$$= \frac{[aq/\rho_1, aq/\rho_2; q]_\infty}{[aq, aq/\rho_1\rho_2; q]_\infty} \sum_{n=0}^{\infty} \frac{[\rho_1, \rho_2; q]_n (aq/\rho_1\rho_2)^n \alpha_n}{[aq/\rho_1, aq/\rho_2; q]_n}. \tag{1.6}$$

As $\rho_1, \rho_2 \to \infty$, Theorem (1.6) takes the following form

Theorem: If $\beta_n = \dfrac{1}{[q, aq; q]_n} \sum_{r=0}^{n} \dfrac{(-)^r q^{(1+n)r}[q^{-n}; q]_r \alpha_r}{q^{r(r+1)/2}[aq^{1+n}; q]_r}$

Then

$$\sum_{r=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{[aq; q]_\infty} \sum_{n=0}^{\infty} a^n q^{n^2} \alpha_n. \tag{1.7}$$

(ii) Choosing

$$\delta_r = \frac{[\rho_1, \rho_2; q]_r (aq/\rho_1\rho_2)^r [aq/\rho_1\rho_2; q]_{m-r}}{[aq/\rho_1, aq/\rho_2; q]_m [q; q]_{m-r}} \tag{1.8}$$

in (1.4) we have

$$\gamma_n = \frac{[\rho_1, \rho_2; q]_n (aq/\rho_1\rho_2)^n [aq/\rho_1\rho_2; q]_{m-n}}{[aq/\rho_1, aq/\rho_2; q]_m [aq; q]_{2n} [q; q]_{m-n}} \times$$

$${}_3\Phi_2 \left[\begin{matrix} \rho_1 q^n, \rho_2 q^n, q^{-m+n}; q; q \\ aq^{1+2n}, \dfrac{\rho_1\rho_2}{a} q^{-m+n} \end{matrix} \right]. \tag{1.9}$$

Summing the ${}_3\Phi_2$ series by Saalsch*ü*tz theorem we have

$$\gamma_n = \frac{[\rho_1, \rho_2; q]_n (aq/\rho_1\rho_2)^n}{[aq/\rho_1, aq/\rho_2; q]_n [q; q]_{m-n} [aq; q]_{m+n}}. \tag{1.10}$$

Putting the conjugate Bailey pair (1.8) and (1.10) in (1.2) we get the following theorem

Theorem: If

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q; q]_{n-r} [aq; q]_{n+r}}$$

Then

$$\sum_{n=0}^{m} \frac{[\rho_1, \rho_2; q]_n (aq/\rho_1\rho_2)^n \alpha_n}{[aq/\rho_1, aq/\rho_2; q]_n [q; q]_{m-n} [aq; q]_{m+n}}$$

$$= \frac{1}{[aq/\rho_1, aq/\rho_2; q]_m} \sum_{n=0}^{m} \frac{[\rho_1, \rho_2; q]_n (aq/\rho_1\rho_2)^n [aq/\rho_1\rho_2; q]_{m-n} \beta_n}{[q; q]_{m-n}} \tag{1.11}$$

As $\rho_1 \rho_2 \to \infty$, theorem (1.11) takes the form

Theorem: If

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q; q]_{n-r} [aq; q]_{n+r}}$$

Then

$$\sum_{n=0}^{m} \frac{a^n q^{n^2} \alpha_n}{[q;q]_{m-n}[aq;q]_{m+n}} = \sum_{n=0}^{m} \frac{a^n q^{n^2} \beta_n}{[q;q]_{m-n}}. \tag{1.12}$$

As $m \to \infty$, (1.12) yields (1.7).

(iii) Again, choosing $u_r = \frac{[k/a;q]_r}{[q;q]_r}$, $v_r = \frac{[k;q]_r}{[aq;q]_r}$ then Bailey lemma (1.2) takes the form,

If

$$\beta_n = \sum_{r=0}^{n} \frac{[k/a;q]_{n-r}[k;q]_{n+r}}{[q;q]_{n-r}[aq;q]_{n+r}} \alpha_r \tag{1.13}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \frac{[k/a;q]_r [k;q]_{2n+r}}{[q;q]_r [aq;q]_{2n+r}} \delta_{r+n}, \tag{1.14}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.15}$$

Sequences $\langle \alpha_n, \beta_n \rangle$ satisfying (1.13) are called WP-Bailey pair whereas $\langle \gamma_n, \delta_n \rangle$ satisfying (1.14) are called conjugate WP-Bailey pair. For WP-Bailey pair one is referred to [1,2].

If we choose $\delta_r = \left(\frac{a^2 q}{k^2}\right)^r$ in (1.14) then we have

$$\gamma_n = \frac{[a^2q/k, aq/k;q]_\infty [k;q]_{2n}}{[aq, a^2q/k^2;q]_\infty [a^2q/k;q]_{2n}} \left(\frac{a^2q}{k^2}\right)^n, \tag{1.16}$$

provided $|a^2q/k^2| < 1$.

Putting the values of conjugate WP-Bailey pair (1.16) in (1.15) we get the following theorem,

Theorem: If

$$\beta_n = \sum_{r=0}^{n} \frac{[k/a;q]_{n-r}[k;q]_{n+r}}{[q;q]_{n-r}[aq;q]_{n+r}} \alpha_r$$

Then

$$\sum_{n=0}^{\infty} \frac{[k;q]_{2n}}{[a^2q/k;q]_{2n}} \left(\frac{a^2q}{k^2}\right)^n \alpha_n = \frac{[aq, a^2q/k^2;q]_\infty}{[aq/k, a^2q/k;q]_\infty} \sum_{n=0}^{\infty} \left(\frac{a^2q}{k^2}\right)^n \beta_n. \tag{1.17}$$

(iv) If we take $\delta_r = \left(\frac{a^2}{k^2}\right)^r$ in (1.14) then we have

$$\gamma_n = \frac{[k;q]_{2n}}{[aq;q]_{2n}} \left(\frac{a^2}{k^2}\right)^n {}_2\Phi_1 \left[\begin{matrix} k/a, kq^{2n}; q; \frac{a^2}{k^2} \\ aq^{1+2n} \end{matrix} \right].$$

Summing the ${}_2\Phi_1$ series by using the summation formula,

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; \frac{c}{ab} \\ cq \end{matrix} \right] = \frac{[cq/a, cq/b; q]_\infty}{[cq, cq/ab; q]_\infty} \left\{ \frac{ab(1+c) - c(a+b)}{ab - c} \right\} \tag{1.18}$$

[3;(3.4) p. 406]

we have

$$\gamma_n = \frac{[aq/k, a^2q/k; q]_\infty}{[aq, a^2q/k^2; q]_\infty} \left(\frac{1}{k+a}\right) \frac{[k;q]_{2n}(1+aq^{2n}) \left(\frac{a^2}{k^2}\right)^n}{[a^2q/k;q]_{2n}} \tag{1.19}$$

Putting these values in (1.15) we have,

Theorem: If

$$\beta_n = \sum_{r=0}^{n} \frac{[k/a;q]_{n-r}[k;q]_{n+r}}{[q;q]_{n-r}[aq;q]_{n+r}} \alpha_r$$

Then

$$\sum_{n=0}^{\infty} \frac{[k;q]_{2n}(1+aq^{2n})}{[a^2q/k;q]_{2n}} \left(\frac{a^2}{k^2}\right)^n \alpha_n$$

$$= \left(\frac{k+a}{k}\right) \frac{[aq, a^2q/k^2; q]_\infty}{[aq/k, a^2q/k; q]_\infty} \sum_{n=0}^{\infty} \left(\frac{a^2}{k^2}\right) \beta_n. \tag{1.20}$$

We shall make use of theorems (1.17) and (1.20) in order to establish transformation formulae for q-series

Following summation formulae are needed in our analysis,

$${}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, kq^n, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq^{1-n}/k, aq^{1+n} \end{matrix} \right] = \frac{[aq, kb/a; q]_n}{[k/a, aq/b; q]_n b^n} \tag{1.21}$$

which can be deduced from [Gasper and Rahman 4; App. II (II.21)]

$${}_8\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2q/bck, kq^n, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bck/a, aq^{1-n}/k, aq^{1+n} \end{matrix} \right] = \frac{[aq, aq/bc, kb/a, kc/a; q]_n}{[aq/c, k/a, aq/b, kbc/a; q]_n}, \tag{1.22}$$

which can be deduced from [Gasper and Rahman 4; App. II (II.22)]

2. WP-Bailey pairs

In this section we shall make use of the following relation in order to evaluate β_n for some proper choice of α_n,

$$\beta_n = \sum_{r=0}^{n} \frac{[k/a;q]_{n-r}[k;q]_{n+r}}{[q;q]_{n-r}[aq;q]_{n+r}}\alpha_r$$

$$= \frac{[k/a;q]_n[k;q]_n}{[q;q]_n[aq;q]_n} \sum_{r=0}^{n} \frac{[q^{-n},kq^n;q]_r}{[aq^{1-n}/k,aq^{1+n};q]_r}\left(\frac{aq}{k}\right)^r \alpha_r. \tag{2.1}$$

(i) Choosing $\alpha_r = \frac{[a,q\sqrt{a},-q\sqrt{a},b;q]_r}{[q,\sqrt{a},-\sqrt{a},aq/b;q]_r}\left(\frac{1}{b}\right)^r$ in (2.1) and using the summation formula (1.21) we get $\beta_n = \frac{[k,kb/a;q]_n}{[q,aq/b;q]_n b^n}$.

Thus,

$$\alpha_n = \frac{[a,q\sqrt{a},-q\sqrt{a},b;q]_n}{[q,\sqrt{a},-\sqrt{a},aq/b;q]_n}\left(\frac{1}{b}\right)^n$$

and

$$\beta_n = \frac{[k,kb/a;q]_n}{[q,aq/b;q]_n b^n} \tag{2.2}$$

are WP-Bailey pair.

(ii) Choosing $\alpha_r = \frac{[a,q\sqrt{a},-q\sqrt{a},b,c,a^2q/bck;q]_r}{[q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bck/a;q]_r}\left(\frac{k}{a}\right)^r$ in (2.1) and using the summation formula (1.22) we get

$$\beta_n = \frac{[k,aq/bc,kb/a,kc/a;q]_n}{[q,aq/b,aq/c,kbc/a;q]_n}. \tag{2.3}$$

α_n, β_n given in (2.3) are WP-Bailey pair.

(iii) Choosing $\alpha_n = \frac{[a,q\sqrt{a},-q\sqrt{a},a/k;q]_n}{[q,\sqrt{a},-\sqrt{a},kq;q]_n}\left(\frac{k}{a}\right)^n$

Then

$$\beta_n = \delta_{n,0} \tag{2.4}$$

α_n, β_n given in (2.4) are WP Bailey pair.

This Bailey pair can be deduced from (2.2).

(iv) From (2.1) we have $\alpha_n = \delta_{n,0}$

Then

$$\beta_n = \frac{[k,k/a;q]_n}{[q,aq;q]_n}. \tag{2.5}$$

α_n, β_n given in (2.5) are also WP Bailey pair.

3. Transformation Formulae

In this section we shall establish transformation formulae for q-series by making use of (1.17), (1.20) and WP-Bailey pairs mentioned in section 2.

(i) Using the WP-Bailey pair (2.2) in (1.17) we get,

$$\sum_{n=0}^{\infty} \frac{[k;q^2]_n[kq;q^2]_n[a,q\sqrt{a},-q\sqrt{a},b;q]_n \left(\frac{a^2q}{k^2b}\right)^n}{[a^2q/k;q^2]_n[a^2q^2/k;q^2]_n[q,\sqrt{a},-\sqrt{a},aq/b;q]_n}$$

$$= \frac{[aq,a^2q/k^2;q]_\infty}{[aq/k,a^2q/k;q]_\infty} \sum_{n=0}^{\infty} \frac{[k,kb/a;q]_n}{[q,aq/b;q]_n} \left(\frac{a^2q}{k^2b}\right)^n,$$

which can be written as,

$${}_8\Phi_7\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},b,\sqrt{k},-\sqrt{k},\sqrt{kq},-\sqrt{kq};q;\dfrac{a^2q}{k^2b} \\ \sqrt{a},-\sqrt{a},aq/b,aq/\sqrt{k},-aq/\sqrt{k},a\sqrt{q/k},-a\sqrt{q/k} \end{matrix}\right]$$

$$= \frac{[aq,a^2q/k^2;q]_\infty}{[aq/k,a^2q/k;q]_\infty}\, {}_2\Phi_1\left[\begin{matrix} k,kb/a;q;\dfrac{a^2q}{k^2b} \\ aq/b \end{matrix}\right], \tag{3.1}$$

provided $\left|\frac{a^2q}{k^2b}\right| < 1$.

(ii) Using the WP-Bailey pair (2.2) in (1.20) we get,

$$\sum_{n=0}^{\infty} \frac{[k;q^2]_n[kq;q^2]_n[a,q\sqrt{a},-q\sqrt{a},b;q]_n(1+aq^{2n})}{[a^2q/k;q^2]_n[a^2q^2/k;q^2]_n[q,\sqrt{a},-\sqrt{a},aq/b;q]_n} \left(\frac{a^2}{k^2b}\right)^n$$

$$= \left(\frac{k+a}{k}\right) \frac{[aq,a^2q/k^2;q]_\infty}{[aq/k,a^2q/k;q]_\infty} \sum_{n=0}^{\infty} \left(\frac{a^2}{bk^2}\right)^n \frac{[k,kb/a;q]_n}{[q,aq/b;q]_n},$$

which can be written as

$${}_{10}\Phi_9\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},b,iq\sqrt{a},-iq\sqrt{a},\sqrt{k},-\sqrt{k},\sqrt{kq},-\sqrt{kq};q;\dfrac{a^2}{k^2b} \\ \sqrt{a},-\sqrt{a},aq/b,i\sqrt{a},-i\sqrt{a},\dfrac{aq}{\sqrt{k}},-\dfrac{aq}{\sqrt{k}},a\sqrt{\dfrac{q}{k}},-a\sqrt{\dfrac{q}{k}} \end{matrix}\right]$$

$$= \frac{[aq, a^2/k^2; q]_\infty}{(1+a)[a/k, a^2q/k; q]_\infty} \, {}_2\Phi_1 \left[\begin{array}{l} k, kb/a; q; \dfrac{a^2}{k^2 b} \\ aq/b \end{array} \right], \tag{3.2}$$

provided $\left| \dfrac{a^2 q}{k^2 b} \right| < 1$.

(iii) Using the WP-Bailey pair (2.3) in (1.17) we get,

$$\sum_{n=0}^{\infty} \frac{[k, kq; q^2]_n [a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2q/bck; q]_n}{\left[\dfrac{a^2q}{k}, \dfrac{a^2q^2}{k}; q^2\right]_n [q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bck/a; q]_n} \left(\frac{aq}{k}\right)^n$$

$$= \frac{[aq, a^2q/k^2; q]_\infty}{[aq/k, a^2q/k; q]_\infty} \sum_{n=0}^{\infty} \frac{[k, aq/bc, kb/a, kc/a; q]_n}{[q, aq/b, aq/c, kbc/a; q]_n} \left(\frac{a^2q}{k^2}\right)^n,$$

which can be written as

$${}_{10}\Phi_9 \left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2q/bck, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{aq}{k} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bck/a, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}} \end{array} \right]$$

$$= \frac{[aq, a^2q/k^2; q]_\infty}{[aq/k, a^2q/k; q]_\infty} \, {}_4\Phi_3 \left[\begin{array}{l} k, aq/bc, kb/a, kc/a; q; \dfrac{a^2q}{k^2} \\ aq/b, aq/c, bck/a \end{array} \right], \tag{3.3}$$

provided $\left| \dfrac{aq}{k} \right| < 1$.

Again, using the WP-Bailey pair (2.3) in (1.20) we have,

$$\sum_{n=0}^{\infty} \frac{[k, kq; q^2]_n [a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2q/bck; q]_n (1+aq^{2n})}{\left[\dfrac{a^2q}{k}, \dfrac{a^2q^2}{k}; q^2\right]_n [q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bck/a; q]_n} \left(\frac{a}{k}\right)^n$$

$$= \left(\frac{k+a}{k}\right) \frac{[aq, a^2q/k^2; q]_\infty}{[aq/k, a^2q/k; q]_\infty} \sum_{n=0}^{\infty} \frac{[k, aq/bc, kb/a, kc/a; q]_n}{[q, aq/b, aq/c, kbc/a; q]_n} \left(\frac{a^2}{k^2}\right)^n,$$

which can be written as

$${}_{12}\Phi_{11} \left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, \dfrac{a^2q}{bck}, iq\sqrt{a}, -iq\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{a}{k} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b}, \dfrac{aq}{c}, \dfrac{bck}{a}, i\sqrt{a}, -i\sqrt{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}} \end{array} \right]$$

$$= \left(\frac{k+a}{k(1+a)}\right) \frac{[aq, a^2q/k^2; q]_\infty}{[aq/k, a^2q/k; q]_\infty} \, {}_4\Phi_3 \left[\begin{matrix} k, aq/bc, kb/a, kc/a; q; \dfrac{a^2}{k^2} \\ aq/b, aq/c, bck/a \end{matrix} \right], \tag{3.4}$$

provided $\left|\dfrac{a}{k}\right| < 1$.

(iv) Using the WP-Bailey pair (2.4) in (1.17) we get

$$\sum_{n=0}^{\infty} \frac{[k, kq; q^2]_n [a, q\sqrt{a}, -q\sqrt{a}, b, c, a/k; q]_n}{\left[\dfrac{a^2q}{k}, \dfrac{a^2q^2}{k}; q^2\right]_n [q, \sqrt{a}, -\sqrt{a}, kq; q]_n} \left(\frac{aq}{k}\right)^n$$

$$= \frac{[aq, a^2q/k^2; q]_\infty}{[aq/k, a^2q/k; q]_\infty},$$

which can be written as

$${}_8\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a/k, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{aq}{k} \\ \sqrt{a}, -\sqrt{a}, kq, aq/\sqrt{k}, -aq/\sqrt{k}, a\sqrt{q/k}, -a\sqrt{q/k} \end{matrix} \right]$$

$$= \frac{[aq, a^2q/k^2; q]_\infty}{[aq/k, a^2q/k; q]_\infty}, \tag{3.5}$$

provided $\left|\dfrac{aq}{k}\right| < 1$.

(v) Using the WP-Bailey pair (2.4) in (1.20) we get

$$\sum_{n=0}^{\infty} \frac{[k, kq; q^2]_n [a, q\sqrt{a}, -q\sqrt{a}, a/k; q]_n (1+aq^{2n})}{[a^2q/k, a^2q^2/k; q^2]_n [q, \sqrt{a}, -\sqrt{a}, kq; q]_n (1+a)} \left(\frac{a}{k}\right)^n$$

$$= \left\{\frac{k+a}{k(1+a)}\right\} \frac{[aq, a^2q/k^2; q]_\infty}{[aq/k, a^2q/k; q]_\infty},$$

which can be written as

$${}_{10}\Phi_9 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a/k, iq\sqrt{a}, -iq\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{a}{k} \\ \sqrt{a}, -\sqrt{a}, kq, i\sqrt{a}, -i\sqrt{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}} \end{matrix} \right]$$

$$= \frac{1}{(1+a)} \frac{[aq, a^2/k^2; q]_\infty}{[a/k, a^2q/k; q]_\infty} \tag{3.6}$$

provided $\left|\frac{a}{k}\right| < 1.$

(vi) Using the WP-Bailey pair (2.5) in (1.20) we get

$$
{}_2\Phi_1\left[\begin{array}{l} k, k/a; q; \dfrac{a^2}{k^2} \\ aq \end{array}\right] = \frac{(1+a)k}{(k+a)}\frac{[aq/k, a^2q/k; q]_\infty}{[aq, a^2q/k^2; q]_\infty}, \tag{3.7}
$$

provided $\left|\frac{a^2}{k^2}\right| < 1.$

Acknowledgement

The first author is thankful to University Grants Commission, New Delhi for sanctioning the minor research project no F. No. 8-3 (115)/2011(MRP/NRCB) under which this work has been done.

References

[1] Andrews, G. E., Bailey's transform, lemma, chains and tree, in special functions 2000: Current perspective and future directions, pp. 1-22, J. Bustone et. al. eds., (Kluwer Academic Publishers, Dordrecht, 2001).

[2] Andrews, G. E. and Berkovich, A., The WP-Bailey tree and its applications, J. London Math. Soc. (2) 66 (2002), 529-549.

[3] Denis, R.Y., Singh, S.N., Singh, S.P. and S. Nidhi, Application of Bailey's pair to q-identities, Bull. Cal. Math. Soc., 103, (5) (2011), 403-412.

[4] Gasper, G. and Rahman, M., Basic Hypergeometric Series, Cambridge University Press, Cambridge 1990.