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EXPANSION FORMULAE INVOLVING THE MULTIVARIABLE I-FUNCTION

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Abstract: Kant et al [2] have given several expansions formulae concerning the multivariable H-function. In this paper, we will give six results concerning the expansion formulas involving the multivariable I-function defined by Prasad [5].

Keywords and Phrases: Multivariable I-function, generalized hypergeometric function, expansion formulae, Laplace transform, inverse Laplace transform.

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1. Introduction and preliminaries

As explained by Srivastava [6], linearization relations of the Clebsch-Gordan involving the sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ and $\{q_n(x)\}_{n=0}^{\infty}$ and their generalizations play an important role in various physical situations. Motivated by the usefulness of such results, Srivastava presented a unified study of various classes of polynomials expansions and multiplication theorems involving the generalized Kampe de Fériet function of several variables. As applications of his results, Srivastava [6] provided extensions of various Clebsch-Gordan type and Niukkanen type linearization relations involving products of several Jacobi and Laguerre polynomials. Inspired by the usefulness of the above mentioned results and works of Kant et al [2], we aim to provide further generalizations of these results to the case of the multivariable I-function defined by Prasad [5]. The results established here are expected to be useful in various physical situations.

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral

$$I(z_1,...,z_r) = I^{0,n_2;0,n_3;...;0,n_r:m',n';...;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;...;p_r,q_r:p',q';...;p^{(r)},q^{(r)}} \left(\begin{array}{c|c} z_1 & (a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};...; \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};...; \end{array} \right.$$

$$\left.\begin{matrix}(a_{rj};\alpha'_{rj},...,\alpha^{(r)}_{rj})_{1,p_r}:(a^{(1)}_j,\alpha'^{(1)}_j)_{1,p'};...;(a^{(r)}_j,\alpha'^{(r)}_j)_{1,p^{(r)}}\\ \\ (b_{rj};\beta'_{rj},...,\beta^{(r)}_{rj})_{1,q_r}:(b^{(1)}_j,\beta'^{(1)}_j)_{1,q'};...;(b^{(r)}_j,\beta'^{(r)}_j)_{1,q^{(r)}}\end{matrix}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}...\int_{L_r}\xi(s_1,...,s_r)\prod_{i=1}^{r}\phi_i(s_i)z_i^{s_i}ds_1...ds_r \tag{1.2}$$

The defined integral of the above function, the existence and convergence conditions, see Y.N. Prasad [5].
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as: $|\arg\ z_i| < \frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}}\alpha'^{(i)}_k - \sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha'^{(i)}_k + \sum_{k=1}^{m^{(i)}}\beta'^{(i)}_k - \sum_{k=m^{(i)}+1}^{q^{(i)}}\beta'^{(i)}_k + \left(\sum_{k=1}^{n_2}\alpha^{(i)}_{2k} - \sum_{k=n_2+1}^{p_2}\alpha^{(i)}_{2k}\right) +$$

$$+\left(\sum_{k=1}^{n_r}\alpha^{(i)}_{rk} - \sum_{k=n_r+1}^{p_r}\alpha^{(i)}_{rk}\right) - \left(\sum_{k=1}^{q_2}\beta^{(i)}_{2k} + \sum_{k=1}^{q_3}\beta^{(i)}_{3k} + ... + \sum_{k=1}^{q_r}\beta^{(i)}_{rk}\right) \tag{1.3}$$

where $i = 1, ..., r$.
The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.
We may establish the the asymptotic expansion in the following convenient form:

$$I(z_1,...,z_r) = 0(|z_1|^{\alpha'_1},...,|z_r|^{\alpha'_r}), \max(|z_1|,...,|z_r|) \to 0$$

$$I(z_1,...,z_r) = 0(|z_1|^{\beta'_1},...,|z_r|^{\beta'_r}), \min(|z_1|,...,|z_r|) \to \infty$$

where $k = 1, ..., z$; $\alpha''_k = \min[Re(b^{(k)}_j/\beta^{(k)}_j)], j = 1, ..., m_k$ and $\beta''_k = \max[Re((a^{(k)}_j - 1)/\alpha^{(k)}_j)]$, $j = 1, ..., n_k$. Throughout this paper, we will note. Let,

$$U = p_2, q_2; p_3, q_3; ...; p_{r-1}, q_{r-1} \tag{1.4}$$

$$V = 0, n_2; 0, n_3; ...; 0, n_{r-1} \tag{1.5}$$

$$X = m^{(1)}, n^{(1)}; ...; m^{(r)}, n^{(r)} \tag{1.6}$$

$$Y = p^{(1)}, q^{(1)}; ...; p^{(r)}, q^{(r)} \tag{1.7}$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)})_{1,p_2}; ...; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, ..., \alpha_{(r-1)k}^{(r-1)})_{1,p_{r-1}} \tag{1.8}$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)})_{1,q_2}; ...; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, ..., \beta_{(r-1)k}^{(r-1)})_{1,q_{r-1}} \tag{1.9}$$

$$\mathbb{A} = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, ..., \alpha_{rk}^{r})_{1,p_r} \tag{1.10}$$

$$\mathbb{B} = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, ..., \beta_{rk}^{r})_{1,q_r} \tag{1.11}$$

$$A' = (a_k^{(1)}, \alpha_k'^{(1)})_{1,p^{(1)}}; ...; (a_k^{(r)}, \alpha_k'^{(r)})_{1,p^{(r)}} \tag{1.12}$$

$$B' = (b_k^{(1)}, \beta_k'^{(1)})_{1,q^{(1)}}; ...; (b_k^{(r)}, \beta_k'^{(r)})_{1,q^{(r)}} \tag{1.13}$$

We have

$$I(z_1, ..., z_r) = I_{U;p_r,q_r;Y}^{V;0,n_r;X} \left(\begin{array}{c|c} z_1 & A; \mathbb{A} : A' \\ . & \\ . & \\ . & \\ z_r & B; \mathbb{B} : B' \end{array} \right) \tag{1.14}$$

2. Required Results

The following two integrals are required in the sequel [3, page 59-60, eqs. 3.6(13), 3.6(19)]

$$\int_0^\infty e^{-zt} t^{\sigma-1} {}_pF_q \left(\begin{array}{c} (\alpha_p) \\ ... \\ (\beta_q) \end{array} ; wt \right) dt = \frac{\Gamma(\sigma)}{z^\sigma} {}_{p+1}F_q \left(\begin{array}{c} \sigma, (\alpha_p) \\ ... \\ (\beta_q) \end{array} ; \frac{w}{z} \right) \tag{2.1}$$

provided: $p \leq q, Re(\sigma) > 0$; $Re(z) > 0$ if $p < q$ and $Re(z) = Re(w)$ if $p = q$

$$\frac{\Gamma(\beta)}{2\pi\omega} \int_{c-i\omega}^{c+i\omega} e^{wt} t^{-\beta} {}_pF_q \left(\begin{array}{c} (\alpha_p) \\ ... \\ (\beta_q) \end{array} ; \frac{z}{t} \right) dt = w^{\beta-1} {}_pF_{q+1} \left(\begin{array}{c} (\alpha_p) \\ ... \\ \beta, (\beta_q) \end{array} ; wz \right) \tag{2.2}$$

provided: $p < q + 1$, $Re(\beta) > 0$.

We also require the following expansion formula involving the polynomial $D_m^{(\alpha)}(x)$ given by

$$D_m^{(\alpha)}(x) = {}_pF_q \left(\begin{array}{c} -m, 1 + \beta(1-\alpha)^{-1}, (\alpha_p) \\ 1 + \beta - m\alpha, \beta(1-\alpha)^{-1}, (\beta_q) \end{array} ; x \right)$$

[see 4, page 156, eq (4.18)]

We have

Lemme 1 For $\alpha > 0, \alpha \neq 1$

$$x^{\sigma}\ {}_kF_s[(g_k);(h_s);ax] = \sum_{m,n=0}^{\infty} \beta \frac{\Gamma(1+\sigma+n)\Gamma(\beta+\sigma-m\alpha+n)\prod_{j=1}^{q}\Gamma(\sigma+\beta_j+n)}{\Gamma(1-m\alpha+\beta)\Gamma(1+\sigma-m+n)\prod_{j=1}^{p}\Gamma(\sigma+\alpha_j+n)}$$

$$D_m^{(\alpha)}(x) f[n;(\alpha_p),(\beta_q);(g_k),(h_s)] \frac{(-)^m a^n}{m!n!} \tag{2.3}$$

where

$$f[n;(\alpha_p),(\beta_q);(g_k),(h_s)] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)\prod_{j=1}^{k}(g_j)_n}{\prod_{j=1}^{q}\Gamma(\beta_j)\prod_{j=1}^{s}(h_j)_n} \tag{2.4}$$

Provided that the double series on the right of (2.3) are absolutely convergent. The polynomials $D_m^{(\alpha)}(x)$ provides a generalization of Laguerre polynomials. We have

Lemme 2

For $\alpha_i > 0, i = 1, ..., n$

$$u^{\alpha_1}...u^{\alpha_n} = \prod_{i=1}^{n}\left\{\Gamma(\alpha_j+1)\Omega[\beta_p^{(i)};(v_q^{(i)})]\frac{\prod_{j=1}^{p}\Gamma(v_j^{(i)}+\alpha_i)}{\prod_{j=1}^{q}\Gamma(\beta_j^{(i)}+\alpha_i)}\right\}\sum_{m=0}^{\infty}\prod_{i=1}^{n}\times$$

$$\left\{\frac{(-)^m}{m!}\frac{(2m+\gamma_i)\Gamma(m+\gamma_i)}{\Gamma(\alpha_i-m+1)\Gamma(\alpha_i+\gamma_i+m+1)}\ {}_{p+2}F_q\left(\begin{matrix}-m, m+\gamma_i,(\beta_p^{(i)}) \\ ... \\ (v_q^{(i)})\end{matrix}\ ;u_i\right)\right\} \tag{2.5}$$

where

$$\Omega[(\beta_p^{(i)}),(v_q^{(i)})] = \frac{\prod_{j=1}^{p}\Gamma(\beta_j^{(i)})}{\prod_{j=1}^{q}\Gamma(v_j^{(i)})}, \quad i = 1, ..., n \tag{2.6}$$

provided that $Re(\gamma_i - v_l^{(i)}) > -1$, $Re(\beta_j^{(i)}) > 0$, $Re(v_j^{(i)}) > 0$, $0 < u_i < 1$; $i = 1, ..., n$; $j = 1, ..., p$; $l = 1, ..., q$.

3. Expansions Formulae

Let

$$A_k = u_1^{\alpha_k^{(1)}} ... u_n^{\alpha_k^{(n)}}, \alpha_k^{(i)} \geq 0; k = 1, ..., r, i = 1, ..., n \tag{3.1}$$

also let

$$\Omega[(\beta_p^{(i)}),(v_q^{(i)})]=\frac{\prod_{j=1}^{p}\Gamma(\beta_j^{(i)})}{\prod_{j=1}^{p}\Gamma(v_j^{(i)})},\quad i=1,...,n \tag{3.2}$$

$$\mathbb{A}_1=\mathbb{A},(-\rho_j;\alpha_1^{(j)},...,\alpha_r^{(j)})_{1,n},(1-\rho_1-v_j^{(1)};\alpha_1^{(1)},...,\alpha_r^{(1)})_{1,q},$$
$$,...,(1-\rho_n-v_j^{(n)};\alpha_1^{(n)},...,\alpha_r^{(n)})_{1,q} \tag{3.3}$$

$$\mathbb{B}_1=\mathbb{B},(-\rho_j+m;\alpha_1^{(j)},...,\alpha_r^{(j)})_{1,n},(-\rho_j-\gamma_j-m;\alpha_1^{(j)},...,\alpha_r^{(j)})_{1,n},$$
$$(1-\rho_j-\beta_j^{(1)};\alpha^{(1)},...,\alpha_r^{(1)})_{1,p},...,(1-\rho_n-\beta_j^{(n)};\alpha_1^{(n)},...,\alpha_r^{(n)})_{1,p} \tag{3.4}$$

Then for the set of extended Jacobi polynomials $D_m(u_i)$, given by

$$D_m(u_i)={}_{p+2}F_q\left(\begin{matrix}-m,m+\gamma_i,(\beta_p^{(i)})\\ ...\\ (v_q^{(i)})\end{matrix};u_i\right),\ i=1,...,n;m\in\mathbb{N} \tag{3.5}$$

We have

Theorem 1

$$u^{\rho_1}...u^{\rho_n}I_{U;p_r,q_r;Y}^{V;0,n_r;X}\left(\begin{matrix}z_1A_1\\ .\\ .\\ .\\ z_rA_r\end{matrix}\left|\begin{matrix}A;\mathbb{A}:A'\\ \\ \\ \\ B;\mathbb{B}:B'\end{matrix}\right.\right)=\prod_{i=1}^{n}\Omega[(\beta_p^{(i)}),(v_q^{(i)})]\sum_{m=0}^{\infty}\prod_{i=1}^{n}\times$$

$$\left\{\frac{(-)^m}{m!}(2m+\gamma_i)\Gamma(m+\gamma_i)D_m(u_i)\right\}I_{U;p_r+n(1+q),q_r+n(2+p);Y}^{V;0,n_r+n(1+q);X}\left(\begin{matrix}z_1\\ .\\ .\\ .\\ z_r\end{matrix}\left|\begin{matrix}A;\mathbb{A}_1:A'\\ \\ \\ \\ B;\mathbb{B}_1:B'\end{matrix}\right.\right) \tag{3.6}$$

provided that $Re(\gamma_i-v_l^{(i)})>-1$, $Re(\beta_j^{(i)})>0$, $Re(v_j^{(i)})>0$, $0<u_i<1$; $i=1,...,n$; $j=1,...,p$; $l=1,...,q$.

Let

$$A_k=u_1^{\alpha_k^{(1)}}...u_n^{\alpha_k^{(n)}},\alpha_k^{(i)}\geq 0;k=1,...,r,i=1,...,n \tag{3.7}$$

also let

$$\Omega[(\beta_p^{(i)}),(v_q^{(i)})]=\frac{\prod_{j=1}^{p}\Gamma(\beta_j^{(i)})}{\prod_{j=1}^{p}\Gamma(v_j^{(i)})},i=1,...,n \tag{3.8}$$

$$\mathbb{A}_1 = \mathbb{A}, (-\rho_j; \alpha_1^{(j)}, ..., \alpha_r^{(j)})_{1,n}, (1-\rho_1 - v_j^{(1)}; \alpha_1^{(1)}, ..., \alpha_r^{(1)})_{1,q},$$
$$, ..., (1-\rho_n - v_j^{(n)}; \alpha_1^{(n)}, ..., \alpha_r^{(n)})_{1,q}. \quad (3.9)$$

$$\mathbb{B}_2 = \mathbb{B}, (-\rho_j + m; \alpha_1^{(j)}, ..., \alpha_r^{(j)})_{1,n}, (1-\rho_j - \beta_j^{(1)}; \alpha_1^{(1)}, ..., \alpha_r^{(1)})_{1,p},$$
$$(1-\rho_n - \beta_j^{(n)}; \alpha_1^{(n)}, ..., \alpha_r^{(n)})_{1,p}. \quad (3.10)$$

Then for the set of extended Jacobi polynomials $D_m^*(u_i)$, given by

$$D_m^*(u_i) = {}_{p+1}F_q \left(\begin{array}{ll} -m, (\beta_p^{(i)}) & \\ ... & ; u_i \\ (v_q^{(i)}) & \end{array} \right), \ i = 1, ..., n; m \in \mathbb{N} \quad (3.11)$$

We have

Theorem 2

$$u^{\rho_1} ... u^{\rho_n} I_{U;p_r,q_r;Y}^{V;0,n_r;X} \left(\begin{array}{c|c} z_1 A_1 & A; \mathbb{A} : A' \\ . & \\ . & \\ . & \\ z_r A_r & B; \mathbb{B} : B' \end{array} \right) = \prod_{i=1}^{n} \Omega[(\beta_p^{(i)}), (v_q^{(i)})] \sum_{m=0}^{\infty} \prod_{i=1}^{n} \times$$

$$\left\{ \frac{(-)^m}{m!} (2m + \gamma_i) \Gamma(m + \gamma_i) D_m^*(u_i) \right\} I_{U;p_r+n(1+q),q_r+n(1+p);Y}^{V;0,n_r+n(1+q);X} \left(\begin{array}{c|c} z_1 & A; \mathbb{A}_1 : A' \\ . & \\ . & \\ . & \\ z_r & B; \mathbb{B}_2 : B' \end{array} \right) \quad (3.12)$$

provided that $Re(\beta_j^{(i)}) > 0$, $Re(v_j^{(i)}) > 0$, $0 < u_i < 1$; $i = 1, ..., n$; $j = 1, ..., p$; $l = 1, ..., q$ and provided that the series on the right of (3.12) is absolutely convergent. The polynomial $D_m^*(u_i)$ is a generalization of Laguerre polynomials.

Let

$$\mathbb{A}_3 = \mathbb{A}, (-\sigma - n; \rho_1, ..., \rho_r), (1-\beta-\sigma+m\alpha-n; \rho_1, ..., \rho_r), (1-\sigma-n-\beta_j; \rho_1, ..., \rho_r)_{1,q} \quad (3.13)$$

$$\mathbb{B}_3 = \mathbb{B}, (-\sigma + m - n; \rho_1, ..., \rho_r), (1 - \sigma - n - \alpha_j; \rho_1, ..., \rho_r)_{1,p} \quad (3.14)$$

Then for the set of polynomials $D_m^{(\alpha)}(u)$ given by

$$D_m^{(x)} = {}_pF_q \left(\begin{array}{ll} -m, 1 + \beta(1-\alpha)^{-1}, (\alpha_p) & \\ ... & ; x \\ 1 + \beta - m\alpha, \beta(1-\alpha)^{-1}, (\beta_q) & \end{array} \right) \quad (3.15)$$

we have

Theorem 3

$$x^{\sigma}\ {}_kF_s[(g_k);(h_s);ax]I^{V;0,n_r;X}_{U;p_r,q_r;Y}\left(\begin{array}{c|c} z_1x^{\rho_1} & A;\mathbb{A}:A' \\ \cdot & \\ \cdot & \\ \cdot & \\ z_rx^{\rho_r} & B;\mathbb{B}:B' \end{array}\right) = \sum_{m,n=0}^{\infty}\frac{(-)^m\beta}{\Gamma(1-m\alpha+\beta)}\times$$

$$f[n;(\alpha_p),(\beta_q);(g_k),(h_s)]\frac{a^n}{m!n!}I^{V;0,n_r+q+2;X}_{U;p_r+q+2,q_r+p+1;Y}\left(\begin{array}{c|c} z_1 & A;\mathbb{A}_3:A' \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & B;\mathbb{B}_3:B' \end{array}\right)D^{(\alpha)}_m(x) \tag{3.16}$$

Provided that the double series on the right of (3.15) are absolutely convergent.

Proofs:

To establish (3.6), we first replace the multivariable I-function defined by Prasad by the Mellin-Barnes contour integral with the help of (1.2), use the Lemme 2 given by (2.5) therein interchange the order of summation and integration (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Finally interpreting the resulting Mellin-Barnes contour integral as the multivariable I-function defined by (1.2), we get the desired formula (3.6).

If in (3.6), we replace z_i by $z_iA_i, i = 1,...,r$ and u_j by $u_j/\gamma_j, j = 1,...,n$ and let $\gamma_j \to \infty$, we obtain (3.12). The proof of (3.16) is similar to (3.6) with the help of Lemme 1.

4. Extensions

In this section, further extensions of the expansion formulae (3.6), (3.12) and (3.16) are provided by making use of the technique of Laplace transforms and inverse Laplace transform by using (2.1) and (2.2) respectively, several times. We have the following results.

Let

$$\mathbb{A}_4 = \mathbb{A},(e^{(1)}_j,\alpha^{(1)}_1,...,\alpha^{(1)}_r)_{1,s},...,(e^{(n)}_j,\alpha^{(n)}_1,...,\alpha^{(n)}_r)_{1,s} \tag{4.1}$$

$$\mathbb{B}_4 = \mathbb{B},(\epsilon^{(1)}_j,\alpha^{(1)}_1,...,\alpha^{(1)}_r)_{1,s},...,(\epsilon^{(n)}_j,\alpha^{(n)}_1,...,\alpha^{(n)}_r)_{1,s} \tag{4.2}$$

Then for the set of extented Jacobi polynomials $D_m(u_i)$, given by

$$B_m(u_i) = {}_{p+s+2}F_{q+k}\left(\begin{array}{l} -m, m+\gamma_i, 1-\rho_i-(e_s^{(i)}), (\beta_p^{(i)}) \\ ... \\ 1-\rho_i-(\epsilon_k^{(i)}), (v_q^{(i)}) \end{array} ; u_i\right), i=1,...,n; m \in \mathbb{N} \tag{4.3}$$

we have

Theorem 4

$$u_1^{\rho_1}...u_n^{\rho_n} I_{U;p_r+ns,q_r+nk;Y}^{V;0,n_r+ns;X}\left(\begin{array}{c|c} z_1 A_1 & A;\mathbb{A}_4 : A' \\ . & \\ . & \\ . & \\ z_r A_r & B;\mathbb{B}_4 : B' \end{array}\right)$$

$$= \prod_{i=1}^{n}\left\{\Omega[(\beta_p^{(i)}),(v_q^{(i)})]\frac{\prod_{j=1}^{s}\Gamma(1-\rho_i-e_j^{(i)})}{\prod_{j=1}^{k}\Gamma(1-\rho_i-\epsilon_j^{(i)})}\right\}\sum_{m=0}^{\infty}\prod_{i=1}^{n}\left\{\frac{(-)^m}{m!}(2m+\gamma_i)\right.$$

$$\left.\Gamma(m+\gamma_i)B_m(u_i)\right\} I_{U;p_r+n(1+q),q_r+n(2+p);Y}^{V;0,n_r+n(1+q);X}\left(\begin{array}{c|c} z_1 & A;\mathbb{A}_1 : A' \\ . & \\ . & \\ . & \\ z_r & B;\mathbb{B}_1 : B' \end{array}\right) \tag{4.4}$$

provided that $Re(\gamma_i - v_l^{(i)}) > -1$, $Re(\beta_j^{(i)}) > 0$, $Re(v_j^{(i)}) > 0$, $0 < u_i < 1$; $i = 1,...,n$; $j = 1,...,p$; $l = 1,...,q$, $t = 1,...,s$ and $Re\left[e_t^{(i)} - \sum_{k=1}^{r}\alpha_k^{(i)}\min_{1\leq j\leq m^{(k)}}\frac{b_j^{(k)}}{\beta_j^{(k)}}\right] < 1$.

For the set polynomials $D_m(u_i)$, given by

$$B_m^*(u_i) = {}_{p+s}F_{q+k}\left(\begin{array}{l} -m, 1-\rho_i-(e_s^{(i)}), (\beta_p^{(i)}) \\ ... \\ 1-\rho_i-(\epsilon_k^{(i)}), (v_q^{(i)}) \end{array} ; u_i\right), \ i=1,...,n; m \in \mathbb{N} \tag{4.5}$$

We have

Theorem 5

$$u_1^{\rho_1}...u_n^{\rho_n} I_{U;p_r+ns,q_r+nk;Y}^{V;0,n_r+ns;X}\left(\begin{array}{c|c} z_1 A_1 & A;\mathbb{A}_4 : A' \\ . & \\ . & \\ . & \\ z_r A_r & B;\mathbb{B}_4 : B' \end{array}\right)$$

$$= \prod_{i=1}^{n} \left\{ \Omega[(\beta_p^{(i)}), (v_q^{(i)})] \frac{\prod_{j=1}^{s} \Gamma(1-\rho_i - e_j^{(i)})}{\prod_{j=1}^{k} \Gamma(1-\rho_i - \epsilon_j^{(i)})} \right\} \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left\{ \frac{(-)^m}{m!} B_m^*(u_i) \right\}$$

$$I_{U;p_r+n(1+q),q_r+n(1+p);Y}^{V;0,n_r+n(1+q);X} \left(\begin{array}{c|c} z_1 & A; \mathbb{A}_1 : A' \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & B; \mathbb{B}_2 : B' \end{array} \right) \tag{4.6}$$

provided that $Re(\beta_j^{(i)}) > 0$, $Re(v_j^{(i)}) > 0$, $0 < u_i < 1$; $i = 1, ..., n$; $j = 1, ..., p$; $l = 1, ..., q$, $t = 1, ..., s$ and $Re \left[e_t^{(i)} - \sum_{k=1}^{r} \alpha_k^{(i)} \min_{1 \leq j \leq m^{(k)}} \frac{b_j^{(k)}}{\beta_j^{(k)}} \right] < 1$.

Let

$$\mathbb{A}_5 = \mathbb{A}, (e_j - n; \rho_1, ..., \rho_r)_{1,p_1} \text{ and } \mathbb{B}_5 = \mathbb{B}, (\epsilon_j - n; \rho_1, ..., \rho_r)_{1,q_1} \tag{4.7}$$

For the set polynomials

$$B_m^{(\alpha)}(x) = {}_{p+p_1+2}F_{q+q_1+2} \left(\begin{array}{l} -m, 1+\beta(1-\alpha)^{-1}, 1-\sigma-(e_{p_1}), (\alpha_p) \\ ... \\ 1+\beta-m\alpha, \beta(1-\alpha)^{-1}, 1-\sigma-(\epsilon_{p_1}), (\beta_q) \end{array} ; x \right) \tag{4.8}$$

We have

Theorem 6

$$x^{\rho} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{k} (g_j)_n (ax)^n}{\prod_{j=1}^{s} (h_j)_n n!} I_{U;p_r+p_1,q_r+q_1;Y}^{V;0,n_r+p_1;X} \left(\begin{array}{c|c} z_1 x^{\rho_1} & A; \mathbb{A}_5 : A' \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r x^{\rho_r} & B; \mathbb{B}_5 : B' \end{array} \right)$$

$$= \frac{\prod_{j=1}^{p_1} \Gamma(1-\sigma-e_j)}{\prod_{j=1}^{q_1} \Gamma(1-\sigma-\epsilon_j)} \sum_{m,n=0}^{\infty} \frac{(-)^m \beta}{\Gamma(1-m\alpha+\beta)} f[n; (\alpha_p), (\beta_q); (g_k), (h_s)]$$

$$I_{U;p_r+q+2,q_r+p+1;Y}^{V;0,n_r+q+2;X} \left(\begin{array}{c|c} z_1 & A; \mathbb{A}_3 : A' \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & B; \mathbb{B}_3 : B' \end{array} \right) B_m^{(\alpha)}(x) \frac{a^n}{m!n!} \tag{4.9}$$

Provided that the double series on the right of (4.9) are absolutely convergent.

Remark:
If $U = V = A = B = 0$, the multivariable I-function defined by Prasad [5] reduces to multivariable H-function defined by Srivastava et al [7], for more details, see Kant et al [2].

5. Conclusion

In this paper we have evaluated the generalized expansions formulae concerning the multivariable I-functions defined by Prasad [5]. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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