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SOME BILINEAR GENERATING RELATIONS INVOLVING CLASSICAL HERMITE POLYNOMIALS VIA MEHLER'S FORMULA

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Abstract: In this paper, using series decomposition technique in Mehler's formula, we obtain some bilinear generating relations associated with classical Hermite's polynomials of even and odd degree.

Keywords and Phrases: Mehler's formula, Classical Hermite's polynomials, Decomposition technique.

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1. Introduction and preliminaries

Throughout in present paper, we use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \backslash \{0\}$. Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases}, \tag{1.1}$$

it being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

The generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters is defined by [16, p.42 Eq.(1)]

$$ {}_pF_q\left[\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}. \tag{1.2}$$

Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that $\beta_j \neq 0, -1, -2, \ldots; \quad j = 1, 2, \ldots, q$.

Supposing that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the usual restriction on β_j, the ${}_pF_q$ series in (1.2)
(i) converges for $|z| < \infty$ if $p \leq q$,
(ii) converges for $|z| < 1$, if $p = q + 1$,
(iii) diverges for all z, $z \neq 0$, if $p > q + 1$.

Furthermore, if we set

$$\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \tag{1.3}$$

it is known that the ${}_pF_q$ series, with $p = q + 1$, is
(I) absolutely convergent for $|z| = 1$, if $\Re(\omega) > 0$,
(II) conditionally convergent for $|z| = 1$, $|z| \neq 1$, if $-1 < \Re(\omega) \leq 0$,
(III) divergent for $|z| = 1$, if $\Re(\omega) \leq -1$.

The idea of separation of a power series into its even and odd terms, exhibited by the elementary identity

$$\sum_{n=0}^{\infty} \Phi(n) = \sum_{n=0}^{\infty} \Phi(2n) + \sum_{n=0}^{\infty} \Phi(2n+1)\ , \tag{1.4}$$

is at least as old as the series themselves. Indeed, when (1.4) is applied to the generalized hypergeometric series (1.2), we are led rather immediately to the elegant result subject to the suitable convergence condition,

$$ {}_pF_q\left[\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z\right] = {}_{2p}F_{2q+1}\left[\begin{array}{c} \frac{\alpha_1}{2}, \frac{1+\alpha_1}{2}, \ldots, \quad \frac{\alpha_p}{2}, \frac{1+\alpha_p}{2}; \\ \frac{1}{2}, \frac{\beta_1}{2}, \frac{1+\beta_1}{2}, \ldots, \frac{\beta_q}{2}, \frac{1+\beta_q}{2}; \end{array} \frac{z^2}{4^{1-p+q}}\right] + $$

$$+z\frac{\alpha_1\ldots\alpha_p}{\beta_1\ldots\beta_q}\,{}_{2p}F_{2q+1}\left[\begin{array}{l}\frac{1+\alpha_1}{2},\frac{2+\alpha_1}{2},\ldots,\quad\frac{1+\alpha_p}{2},\frac{2+\alpha_p}{2};\\ \frac{3}{2},\frac{1+\beta_1}{2},\frac{2+\beta_1}{2},\ldots,\frac{1+\beta_q}{2},\frac{2+\beta_q}{2};\end{array}\frac{z^2}{4^{1-p+q}}\right]. \tag{1.5}$$

The history of the series identity (1.5) is remarkably fascinating. It was given in 1954 by MacRobert [3, p. 95, Eq. (8)] in terms of his E - function.

Ever since over a decade after the publication of MacRobert's paper [3], the hypergeometric series identity (1.5) has been rediscovered a number of times. For instance, it was published in 1966 by Srivastava [15, p.763], in 1969 by Barr [1, p.591, Eq. (1)], in 1970 by Carlson [2, p.234, Eq.(10)], and in 1974 by Manocha [5, p.43, Eq. (3)]. Later on, it has been claimed as a "new" result by Sharma [13, p. 95 Eq. (1)] who apparently rederived it in his supposedly earlier paper [12] which, in fact, has just appeared. {See also [13], p. 99, Line 6.} Except Carlson [2] who did (after having completed the revision of his paper) attribute (1.5) to MacRobert [3], and possibly Barr [1] whose work was indeed motivated by Srivastava's paper [15], these authors do not exhibit their familiarity with the available literature on the hypergeometric series identity(1.5). The second term on the right-hand side of (1.5) was missing in Srivastava's paper [15, p. 763]; in fact, it was subsequently pointed out by Barr [1, p. 592].

The classical Hermite's polynomials $H_n(x)$ are defined by means of the following linear generating relation [11, p.187 eq(1)]

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}, \tag{1.6}$$

which is valid for all finite values of x and t.
Hypergeometric form of classical Hermite polynomials are given below

$$H_n(x)=(2x)^n\,{}_2F_0\left[\begin{array}{c}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};\\ -\quad ;\end{array}-\frac{1}{x^2}\right], \tag{1.7}$$

$$H_{2n}(x)=(2x)^{2n}\,{}_2F_0\left[\begin{array}{c}-n,\ \frac{1}{2}-n\ ;\\ -\quad ;\end{array}-\frac{1}{x^2}\right], \tag{1.8}$$

$$H_{2n+1}(x)=(2x)^{2n+1}\,{}_2F_0\left[\begin{array}{c}-n,-\frac{1}{2}-n;\\ -\quad ;\end{array}-\frac{1}{x^2}\right]. \tag{1.9}$$

If a three-variable function $F(x,y,t)$ possesses a formal power series expansion in t such that

$$F(x,y,t)=\sum_{n=0}^{\infty}\gamma_n f_n(x)f_n(y)t^n,$$

where the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is independent of x, y and t, then $F(x, y, t)$ is called a bilinear generating function for the set $\{f_n(x)\}_{n=0}^{\infty}$.
More generally, if $\mathcal{F}(x, y, t)$ can be expanded in powers of t in the form

$$\mathcal{F}(x,y,t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) f_{\beta(n)}(y) t^n,$$

where $\alpha(n)$ and $\beta(n)$ are the functions of n which are not necessarily equal, we shall still call $\mathcal{F}(x, y, t)$ a bilinear generating function for the set $\{f_n(x)\}_{n=0}^{\infty}$.

F. G. Mehler's formula [11, p. 198 eq(2)]:

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{t^n}{n!} = (1-4t^2)^{-\frac{1}{2}} \exp\left(\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right), \tag{1.10}$$

$$= (1-4t^2)^{-\frac{1}{2}} \exp\left[y^2 - \frac{(y-2xt)^2}{1-4t^2}\right], \tag{1.11}$$

$$= (1-4t^2)^{-\frac{1}{2}} \exp\left(\frac{4xyt}{1-4t^2}\right) \exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right), \tag{1.12}$$

where $|t^2| < \frac{1}{4}$.
In our analysis, we shall apply the equation (1.12).
The hypergeometric form of Binomial expansion $(1-z)^{-a}$ is given by

$$(1-z)^{-a} = {}_1F_0\left[\begin{array}{ccc} a & ; & z \\ - & ; & \end{array}\right] = \sum_{r=0}^{\infty} \frac{(a)_r z^r}{r!}, \tag{1.13}$$

where $|z| < 1$, and $a \neq 0$,-1,-2,-3,....
Motivated by the work of Carlson [2], MacRobert [3-4], Manocha [5], Mohd. et al. [6], Qureshi and Ahmad [7], Qureshi, Quraishi and Pal [8], Qureshi, Yasmeen and Pathan [9] and Sharma [12-14], we shall obtain some bilinear generating relations associated with classical Hermite's polynomials of even and odd degree, using series decomposition formula (1.4).

2. Bilinear Generating Relations (Different Forms)

Any values of parameters and variables leading to the results which do not make sense, are tacitly excluded (suppose x, y are real numbers and $|t| < \frac{1}{4}$), then

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{t^n}{(2n)!} = (1-4t)^{-\frac{1}{2}} \exp\left(\frac{-4t(x^2+y^2)}{1-4t}\right) {}_0F_1\left[\begin{array}{ccc} - & ; & \frac{4x^2y^2t}{(1-4t)^2} \\ \frac{1}{2} & ; & \end{array}\right], \tag{2.1}$$

$$\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{t^n}{(2n+1)!}$$
$$= 4xy(1-4t)^{-\frac{3}{2}} \exp\left(\frac{-4t(x^2+y^2)}{1-4t}\right) {}_0F_1\left[\begin{array}{ccc} - & ; & \frac{4x^2y^2t}{(1-4t)^2} \\ \frac{3}{2} & ; & \end{array}\right], \tag{2.2}$$

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{t^n}{(2n)!} = (1-4t)^{-\frac{1}{2}} \cosh\left(\frac{4xy\sqrt{t}}{1-4t}\right) \times$$
$$\times \left\{ {}_0F_1\left[\begin{array}{ccc} - & ; & \frac{4(x^2+y^2)^2t^2}{(1-4t)^2} \\ \frac{1}{2} & ; & \end{array}\right] - \frac{4(x^2+y^2)t}{1-4t} {}_0F_1\left[\begin{array}{ccc} - & ; & \frac{4(x^2+y^2)^2t^2}{(1-4t)^2} \\ \frac{3}{2} & ; & \end{array}\right]\right\}, \tag{2.3}$$

$$\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{t^n}{(2n+1)!} = (1-4t)^{-\frac{1}{2}} \frac{1}{\sqrt{t}} \sinh\left(\frac{4xy\sqrt{t}}{1-4t}\right) \times$$
$$\times \left\{ {}_0F_1\left[\begin{array}{ccc} - & ; & \frac{4(x^2+y^2)^2t^2}{(1-4t)^2} \\ \frac{1}{2} & ; & \end{array}\right] - \frac{4(x^2+y^2)t}{1-4t} {}_0F_1\left[\begin{array}{ccc} - & ; & \frac{4(x^2+y^2)^2t^2}{(1-4t)^2} \\ \frac{3}{2} & ; & \end{array}\right]\right\}, \tag{2.4}$$

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{t^n}{(2n)!} = \cosh\left(\frac{4xy\sqrt{t}}{1-4t}\right) \exp\left(\frac{-4(x^2+y^2)t}{1-4t}\right)$$
$$\times \left\{ {}_2F_1\left[\begin{array}{cc} \frac{1}{4}, \frac{3}{4}; & 16t^2 \\ \frac{1}{2}; & \end{array}\right] + 2t\ {}_2F_1\left[\begin{array}{cc} \frac{3}{4}, \frac{5}{4}; & 16t^2 \\ \frac{3}{2}; & \end{array}\right]\right\}, \tag{2.5}$$

$$\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{t^n}{(2n+1)!} = \frac{1}{\sqrt{t}} \sinh\left(\frac{4xy\sqrt{t}}{1-4t}\right) \exp\left(\frac{-4(x^2+y^2)t}{1-4t}\right)$$
$$\times \left\{ {}_2F_1\left[\begin{array}{cc} \frac{1}{4}, \frac{3}{4}; & 16t^2 \\ \frac{1}{2}\quad ; & \end{array}\right] + 2t\ {}_2F_1\left[\begin{array}{cc} \frac{3}{4}, \frac{5}{4}; & 16t^2 \\ \frac{3}{2}\quad ; & \end{array}\right]\right\}. \tag{2.6}$$

3. Derivations of Bilinear Generating Relations Using Different Approaches

First approach

Consider Mehler's formula (1.12):

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{t^n}{n!} = (1-4t^2)^{-\frac{1}{2}} \exp\left(\frac{4xyt}{1-4t^2}\right) \exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right), \tag{3.1}$$

$$= (1-4t^2)^{-\frac{1}{2}} \exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right) \sum_{r=0}^{\infty} \frac{\left(\frac{4xyt}{1-4t^2}\right)^r}{r!}. \tag{3.2}$$

Now apply series decomposition identity (1.4), we obtain

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{t^{2n}}{(2n)!} + t\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{t^{2n}}{(2n+1)!}$$

$$= (1-4t^2)^{-\frac{1}{2}} \exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right)\left[\sum_{r=0}^{\infty}\frac{\left(\frac{4xyt}{1-4t^2}\right)^{2r}}{(2r)!} + \sum_{r=0}^{\infty}\frac{\left(\frac{4xyt}{1-4t^2}\right)^{2r+1}}{(2r+1)!}\right]. \tag{3.3}$$

Putting $t = iT$ or $t^2 = -T^2$, we have

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{(-T^2)^n}{(2n)!} + iT\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{(-T^2)^n}{(2n+1)!} = (1+4T^2)^{-\frac{1}{2}}$$

$$\times \exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)\left[\sum_{r=0}^{\infty}\frac{\left(\frac{4xy}{1+4T^2}\right)^{2r}(-T^2)^r}{(2r)!} + \sum_{r=0}^{\infty}\frac{\left(\frac{4xy}{1+4T^2}\right)^{2r+1}(iT)(-T^2)^r}{(2r+1)!}\right]. \tag{3.4}$$

Suppose x and y are real numbers then equating real and imaginary parts, we get

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{(-T^2)^n}{(2n)!} = (1+4T^2)^{-\frac{1}{2}}$$

$$\times \exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)\left[\sum_{r=0}^{\infty}\frac{(16x^2y^2)^r(-T^2)^r}{[(1+4T^2)^2]^r 2^{(2r)}\left(\frac{1}{2}\right)_r r!}\right], \tag{3.5}$$

$$= (1+4T^2)^{-\frac{1}{2}} \exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right) {}_0F_1\left[\begin{matrix} - & ; & \frac{(-16x^2y^2T^2)}{[4(1+4T^2)^2]} \\ \frac{1}{2} & ; & \end{matrix}\right]. \tag{3.6}$$

$$\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{(-T^2)^n}{(2n+1)!} = (1+4T^2)^{-\frac{1}{2}} \exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right) \times$$

$$\times \frac{4xy}{(1+4T^2)}\left[\sum_{r=0}^{\infty}\frac{(16x^2y^2)^r(-T^2)^r}{[(1+4T^2)^2]^r 2^{(2r)}\left(\frac{3}{2}\right)_r r!}\right], \tag{3.7}$$

$$= (1+4T^2)^{-\frac{1}{2}} \exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)\frac{4xy}{(1+4T^2)} {}_0F_1\left[\begin{matrix} - & ; & \frac{(-16x^2y^2T^2)}{[4(1+4T^2)^2]} \\ \frac{3}{2} & ; & \end{matrix}\right]. \tag{3.8}$$

Finally putting $T = i\sqrt{t}$ or $T^2 = -t$ in equations (3.6) and (3.8), we get generating relations (2.1) and (2.2) respectively.

Second approach

Again consider Mehler's formula (1.12) in the form:

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{t^n}{n!} = (1-4t^2)^{-\frac{1}{2}} \exp\left(\frac{4xyt}{1-4t^2}\right) \exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right), \quad (3.9)$$

$$= (1-4t^2)^{-\frac{1}{2}} \exp\left(\frac{4xyt}{1-4t^2}\right) \sum_{r=0}^{\infty} \left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right)^r \frac{1}{r!}. \quad (3.10)$$

Now apply series decomposition identity (1.4), we obtain

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{t^{2n}}{(2n)!} + t\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{t^{2n}}{(2n+1)!}$$

$$= (1-4t^2)^{-\frac{1}{2}} \exp\left(\frac{4xyt}{1-4t^2}\right) \times$$

$$\times \left[\sum_{r=0}^{\infty} \left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right)^{2r} \frac{1}{2r!} - \frac{4(x^2+y^2)t^2}{1-4t^2} \sum_{r=0}^{\infty} \left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right)^{2r} \frac{1}{(2r+1)!}\right]. \quad (3.11)$$

Putting $t = iT$ or $t^2 = -T^2$, we obtain

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{(-T^2)^n}{(2n)!} + iT\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{(-T^2)^n}{(2n+1)!}$$

$$= (1+4T^2)^{-\frac{1}{2}} \exp\left(\frac{i4xyT}{1+4T^2}\right) \times$$

$$\times \left[\sum_{r=0}^{\infty} \left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)^{2r} \frac{1}{2r!} + \frac{4(x^2+y^2)T^2}{1+4T^2} \sum_{r=0}^{\infty} \left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)^{2r} \frac{1}{(2r+1)!}\right]. \quad (3.12)$$

Now equating real and imaginary parts in equation (3.12), we obtain:

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{(-T^2)^n}{(2n)!} = (1+4T^2)^{-\frac{1}{2}} cos\left(\frac{4xyT}{1+4T^2}\right) \left[\sum_{r=0}^{\infty} \left(\frac{16(x^2+y^2)^2T^4}{(1+4T^2)^2}\right)^r\right.$$

$$\left.\frac{1}{2^{2r}r!(\frac{1}{2})_r} + \frac{4(x^2+y^2)T^2}{1+4T^2} \sum_{r=0}^{\infty} \left(\frac{16(x^2+y^2)^2T^4}{(1+4T^2)^2}\right)^r \frac{1}{2^{2r}r!(\frac{3}{2})_r}\right], \quad (3.13)$$

$$\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{(-T^2)^n}{(2n+1)!}$$

$$= (1+4T^2)^{-\frac{1}{2}}\frac{1}{T}sin\left(\frac{4xyT}{1+4T^2}\right)\left[\sum_{r=0}^{\infty}\left(\frac{16(x^2+y^2)^2T^4}{(1+4T^2)^2}\right)^r\right.$$

$$\left.\frac{1}{2^{2r}r!(\frac{1}{2})_r} + \frac{4(x^2+y^2)T^2}{1+4T^2}\sum_{r=0}^{\infty}\left(\frac{16(x^2+y^2)^2T^4}{(1+4T^2)^2}\right)^r\frac{1}{2^{2r}r!(\frac{3}{2})_r}\right]. \tag{3.14}$$

Finally putting $T = i\sqrt{t}$ or $T^2 = -t$ in equations (3.13) and (3.14), we get generating relations (2.3) and (2.4) respectively.

Third approach

Further consider the Mehler's formula in the form:

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{t^n}{n!} = (1-4t^2)^{-\frac{1}{2}}\exp\left(\frac{4xyt}{1-4t^2}\right)\exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right), \tag{3.15}$$

$$= \exp\left(\frac{4xyt}{1-4t^2}\right)\exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right){}_1F_0\left[\begin{matrix}\frac{1}{2};\\ -;\end{matrix}\ 4t^2\right], \tag{3.16}$$

$$= \exp\left(\frac{4xyt}{1-4t^2}\right)\exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right)\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)_r\frac{(4t^2)^r}{r!}, \tag{3.17}$$

where $|4t^2| < 1$.

Now apply series decomposition identity (1.4), we get

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{t^{2n}}{(2n)!} + t\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{t^{2n}}{(2n+1)!} = \exp\left(\frac{4xyt}{1-4t^2}\right)\times$$

$$\times\exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right)\left[\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)_{2r}\frac{(4t^2)^{2r}}{(2r)!} + \sum_{r=0}^{\infty}\left(\frac{1}{2}\right)_{2r+1}\frac{(4t^2)^{2r+1}}{(2r+1)!}\right]. \tag{3.18}$$

Putting $t = iT$ or $t^2 = -T^2$, we have

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{(-T^2)^n}{(2n)!} + iT\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{(-T^2)^n}{(2n+1)!} = \exp\left(\frac{i4xyT}{1+4T^2}\right)\times$$

$$\times\exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)\left[\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)_{2r}\frac{(-4T^2)^{2r}}{(2r)!} + \frac{1}{2}(-4T^2)\sum_{r=0}^{\infty}\left(\frac{3}{2}\right)_{2r}\frac{(-4T^2)^{2r}}{(2r+1)!}\right], \tag{3.19}$$

$$= \exp\left(\frac{i4xyT}{1+4T^2}\right)\exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)\left[\sum_{r=0}^{\infty} 2^{2r}\left(\frac{1}{4}\right)_r\left(\frac{1}{4}+\frac{1}{2}\right)_r \right.$$

$$\left. \frac{(16T^4)^r}{2^{2r}r!\left(\frac{1}{2}\right)_r} - (2T^2)\sum_{r=0}^{\infty} 2^{2r}\left(\frac{3}{4}\right)_r\left(\frac{3}{4}+\frac{1}{2}\right)_r\frac{(16T^4)^r}{2^{2r}r!\left(\frac{3}{2}\right)_r}\right], \tag{3.20}$$

$$= \left\{\cos\left(\frac{4xyT}{1+4T^2}\right) + i\ \sin\left(\frac{4xyT}{1+4T^2}\right)\right\}\times$$

$$\times \exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)\left[{}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{3}{4};\\ \frac{1}{2}\quad;\end{matrix}\ 16T^4\right) - 2T^2{}_2F_1\left(\begin{matrix}\frac{3}{4},\frac{5}{4};\\ \frac{3}{2}\quad;\end{matrix}\ 16T^4\right)\right]. \tag{3.21}$$

Now equating real and imaginary parts in equation (3.21), we obtain

$$\sum_{n=0}^{\infty} H_{2n}(x)H_{2n}(y)\frac{(-T^2)^n}{(2n)!} = \left\{cos\left(\frac{4xyT}{1+4T^2}\right)\right\}\times$$

$$\times \exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)\left[{}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{3}{4};\\ \frac{1}{2}\quad;\end{matrix}\ 16T^4\right) - 2T^2{}_2F_1\left(\begin{matrix}\frac{3}{4},\frac{5}{4};\\ \frac{3}{2}\quad;\end{matrix}\ 16T^4\right)\right], \tag{3.22}$$

$$\sum_{n=0}^{\infty} H_{2n+1}(x)H_{2n+1}(y)\frac{(-T^2)^n}{(2n+1)!} = \frac{1}{T}\left\{sin\left(\frac{4xyT}{1+4T^2}\right)\right\}\times$$

$$\times \exp\left(\frac{4(x^2+y^2)T^2}{1+4T^2}\right)\left[{}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{3}{4};\\ \frac{1}{2}\quad;\end{matrix}\ 16T^4\right) - 2T^2{}_2F_1\left(\begin{matrix}\frac{3}{4},\frac{5}{4};\\ \frac{3}{2}\quad;\end{matrix}\ 16T^4\right)\right]. \tag{3.23}$$

Finally putting $T = i\sqrt{t}$ or $T^2 = -t$ in equations (3.23) and (3.24), we get generating relations (2.5) and (2.6) respectively.

We conclude our present investigation, by observing that several bilinear generating relations can be obtained from known generating relations, in analogous manner.

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